§3.4–3.5 Permutations and Combinations of Multisets

A multiset $M$ is like a set where the members may repeat. Example:

$$M = \{a, a, a, b, c, c, d, d, d, d\}$$

– or we can write –

$$M = \{3 \cdot a, 1 \cdot b, 2 \cdot c, 4 \cdot d\}$$

Definition: repetition numbers of the members

We allow infinite repetition, in which case we would write $\infty \cdot a$.

We can ask

(and we will answer):

Q: How many $r$-permutations, permutations, and $r$-combinations are there of a certain multiset?

Remember the difference:

$r$-permutations order $r$ elements of $M$.

permutations order all elements of $M$.

$r$-combinations choose $r$ elements of $M$.

(A multisubset, in other words!)
§3.4–3.5 Permutations and Combinations of Multisets

Example: \( r \)-permutations when each element of \( M \) has an infinite repetition number:

Example: \( r \)-combinations when each element of \( M \) has an infinite repetition number:

Example: permutations of a finite multiset \( M \):
§3.4 Permutations of a Finite Multiset

Let $M$ have $k$ different types of members with finite repetition numbers $n_1$ through $n_k$ and let $|M| = n = n_1 + \cdots + n_k$.

How many permutations of $S$ are there?

**Proof 1:** Label all the balls uniquely; how many permutations exist? ____ Now ignore the labelings. How many times does the same multiset permutation appear? [What are the symmetries?]

**Proof 2:** How many ways are there to place the balls of type 1? ____ Once they are placed, how many ways are there to place the balls of type 2? ____
§3.4 Finite-Multiset-Permutation Examples

*Example:* How many permutations of the letters in Mississippi are there?

*Example:* In how many ways can we place $n$ labeled objects into $k$ labeled boxes, where each box $B_i$ contains $n_i$ objects and $n_1 + n_2 + \cdots + n_k = n$?

*Example:* What if all the boxes are all the same size and not labeled?
§3.4 Non-attacking Rooks

In chess, a rook is a piece that moves in a column or in a row. Two rooks are said to attack each other if they are in the same row or column.

Q: How many ways are there to place eight non-attacking rooks on an $8 \times 8$ chessboard?

A: There can not be two in the same row, so there must be one in each row. How many ways are there to place a rook in the first row? ______
Once placed, how many ways for the second? __

In all?
§3.4 More Non-attacking Rooks

Q: In how many ways can eight distinguishable rooks be placed? (Eight different colors, perhaps.)

Q: What if there are four yellow rooks, three blue rooks, and one red rook?

Q: What if we wanted to place six indistinguishable rooks on an $8 \times 8$ chessboard?
§3.5 Infinite-Multiset-Combination Examples

The situation: we are looking for the number of $r$-combinations of a multiset

$$M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$ 

Our original interpretation is that of balls and bars:

| an $r$-combination of $M$ (k types of objects) | a permutation of $r$ balls and $k - 1$ bars |

_Example:_ If the bagel shoppe sells plain, poppy-seed, sesame, and everything bagels, in how many ways are there to make a bag of a dozen bagels?

_Example:_ What if there must be at least one of each kind in the dozen?
§3.5 Infinite-Multiset-Combination Examples

Another interpretation of $r$-combinations of $M$ is non-negative integer solutions of

$$x_1 + x_2 + x_3 + x_4 = r.$$  

**Example:** How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

**Example:** How many positive integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?