The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has $\lambda_i$ boxes in row $i$.

(James, Kerber) Create an **abacus diagram** from the boundary of $\lambda$.

Abacus: Function $a : \mathbb{Z} \to \{\bullet, \omega\}$.

(Equivalence class...)
Core partitions

The **hook length** of a box $= \# \text{ boxes below} + \# \text{ boxes to right} + \text{ box}$

$\lambda$ is a $t$-core if no boxes have hook length $t \leftrightarrow t$-flush abacus

**$t$-core partition**

```
 10 6 5 2 1
  7 3  2
  6 2  1
  3
  2
  1
```

**$t$-flush abacus** (in runners)

```
  8  7  6  5
  4  3  2  1
  0  1  2  3
  4  5  6  7
  8  9 10 11
```

**Self-conj. $t$-core partition**

```
 13  9  7  5  3  2  1
  9  5  3  1
  7  3  1
  5  1
  3
  2
  1
```

**Normalized**

```
 8  9 10 11
```

**Balanced**

```
 9 10 11 12
```

**$t$-flush antisymmetric abacus**

```
  7  6  5  4
  3  2  1  0
  1  2  3  4
  5  6  7  8
  9 10 11 12
```

**Antisymmetry about $t/t + 1$.**

(Discuss defining beads, reading off hooks....)
Simultaneity

Of interest: Partitions that are both $s$-core and $t$-core. $(s, t) = 1$

- Abaci that are both $s$-flush and $t$-flush.

There are infinitely many (self-conjugate) $t$-core partitions.

\[
\begin{array}{cccccc}
9 & 6 & 5 & 3 & 2 & 1 \\
5 & 2 & 1 \\
2 \\
1 \\
\end{array}
\]

(Anderson, 2002): 
$\# (s, t)$-core partitions
$\frac{1}{s+t} \left(\begin{array}{c}s+t \\ s \end{array}\right)$

\[
\begin{array}{cccccc}
9 & 6 & 4 & 2 & 1 \\
6 & 3 & 1 \\
4 & 1 \\
2 \\
1 \\
\end{array}
\]

(Ford, Mai, Sze, 2009): 
$\#$ self-conj. $(s, t)$-core partitions
$\left(\begin{array}{c}s'+t' \\ s' \end{array}\right)$

where $s' = \left\lfloor \frac{s}{2} \right\rfloor$ and $t' = \left\lfloor \frac{t}{2} \right\rfloor$
### Core partitions in the literature

**Representation Theory:** (origin)
- Nakayama conjecture, proved by Brauer & Robinson 1947 says \( t \)-cores label \( t \)-blocks of irreducible modular representations for \( S_n \).

**Number Theory:**
- Let \( c_t(n) = \# \) of \( t \)-core partitions of \( n \).
- In 1976, Olsson proved \[ \sum_{n \geq 0} c_t(n)x^n = \prod_{n \geq 1} \frac{(1 - x^{nt})^t}{1 - x^n} \]

**Numerical properties of \( c_t(n) \)?**
- 1996: Granville & Ono proved positivity: \( c_t(n) > 0 \) \( (t \geq 4) \).
- 1999: Stanton conjectured monotonicity: \( c_{t+1}(n) \geq c_t(n) \)
- 2012: R. Nath & I conjectured monotonicity: \( sc_{t+2}(n) \geq sc_t(n) \)

**Modular forms:** g.f. related to Dedekind’s \( \eta \)-fcn, a m.f. of wt. 1/2.

**Group Theory:** By Lascoux 2001, \( t \)-cores \( \leftrightarrow \) coset reps in \( \tilde{S}_t/S_t \)
Group actions on combinatorial objects!!!
Reflection Groups

The combinatorics of groups:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - Although, it is not the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

We will talk about reflection groups. (With nice pictures)

- $W$ is generated by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of relations.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. — and — $(s_i s_j)^\text{power} = \text{id}$.

For example, $w = s_3 s_2 s_1 s_1 s_2 s_4 = s_3 s_2 \text{id} s_2 s_4 = s_3 \text{id} s_4 = s_3 s_4$
Reflection Groups

- The action of multiplying (on the left) by a generator $s$ corresponds to a reflection across a hyperplane $H_s$. \( (s_i^2 = \text{id}) \)

- When the angle between $H_s$ and $H_t$ is $\frac{\pi}{3}$, relation is \((st)^3 = \text{id}\).

- The group depends on the placement of the hyperplanes. \(|S| = 6\).
An infinite reflection group: the affine permutations $\tilde{S}_n$.

- Add a new generator $s_0$ and a new affine hyperplane $H_0$.

Elements generated by $\{s_0, s_1, s_2\}$ correspond to alcoves here.
Combinatorics of affine permutations

Many ways to reference elements in $\tilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.

They all play nicely with each other.
An abacus model for affine permutations

(James and Kerber, 1981) Given an affine permutation \([w_1, \ldots, w_n]\),

- Place integers in \(n\) runners.
- Circled: beads. Empty: gaps
- Create an abacus where each runner has a lowest bead at \(w_i\).

Example: \([-4, -3, 7, 10]\)

- Generators act nicely.
- \(s_i\) interchanges runners \(i \leftrightarrow i + 1\). \((s_1 : 1 \leftrightarrow 2)\)
- \(s_0\) interchanges runners 1 and \(n\) (with shifts) \((s_0 : 1 \leftrightarrow 4)\)
Action of generators on the core partition

- Label the boxes of $\lambda$ with residues.
- $s_i$ acts by adding or removing boxes with residue $i$.

Example. $\lambda = (5, 3, 3, 1, 1)$ is a 4-core.
- has removable 0 boxes
- has addable 1, 2, 3 boxes.

Idea: We can use this to figure out a word for $w$. 
Finding a word for an affine permutation.

Example: The word in $S_4$ corresponding to $\lambda = (6, 4, 4, 2, 2)$:

$$s_1s_0s_2s_1s_3s_2s_0s_3s_1s_0$$

$$\begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
2 & 3 & 0 & 1 \\
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
\end{array} \quad \quad \begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
2 & 3 & 0 & 1 \\
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
\end{array} \quad \quad \begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
2 & 3 & 0 & 1 \\
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
\end{array}$$
The bijection between cores and alcoves
Simultaneous core partitions

How many partitions are both 2-cores and 3-cores? **2**.

How many partitions are both 3-cores and 4-cores? **5**.

How many simultaneous 4/5-cores? **14**.

How many simultaneous 5/6-cores? **42**.

How many simultaneous $n/(n+1)$-cores? $C_n$!

Jaclyn Anderson proved that the number of $s/t$-cores is $\frac{1}{s+t}(\binom{s+t}{s})$.

The number of 3/7-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10} \cdot \frac{10!}{3!7!} = 12$.

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.
Research Questions

★ Can we extend combinatorial interps to other reflection groups?

- Yes! Involves self-conjugate partitions.
- Joint with Brant Jones, James Madison University.
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★ What numerical properties do self-conjugate core partitions have?
  ► There are more (s.c. $t+2$-cores of $n$) than (s.c. $t$-cores of $n$).
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★ Properties of simultaneous core partitions. (Formula: \( \frac{1}{s+t}(\binom{s+t}{s}) \))
   ► **Question.** What is the average size of an \((s, t)\)-core partition?
   ► **Progress:** Answer: \((s + t + 1)(s - 1)(t - 1)/24\). Proof?
   ► **Question:** Is there a core statistic for a \(q\)-analog of \( \frac{1}{s+t}(\binom{s+t}{s}) \)?
   ► **Progress:** \(m\)-Catalan number \(C_3\) through \((3, 3m + 1)\)-cores.
   ► \((s, t)\)-cores \(\leftrightarrow\) certain lattice paths. Statistics galore!

★ Happy to have students who would like to do research!