Catalan Numbers

\[
c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6 \quad c_7 \quad c_8 \quad c_9 \quad c_{10}
\]

\[
1 \quad 1 \quad 2 \quad 5 \quad 14 \quad 42 \quad 132 \quad 429 \quad 1430 \quad 4862 \quad 16796
\]


\[
c_n = \frac{1}{n+1} \binom{2n}{n}.
\]
Catalan Numbers

<table>
<thead>
<tr>
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Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), . . . (a$^8$) to (y$^8$).
Catalan Numbers

\[
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10} = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796
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Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), \ldots (a^8) to (y^8).

- triangulations of an \((n+2)\)-gon
- lattice paths from \((0, 0)\) to \((n, n)\) above \(y = x\)
- sequences with \(n + 1\)'s, \(n - 1\)'s with positive partial sums
- multiplication schemes to multiply \(n + 1\) numbers
Catalan Number Interpretations

When \( n = 3 \), there are \( c_3 = 5 \) members of these families of objects:

1. Triangulations of an \( (n + 2) \)-gon
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3. Sequences of length \(2n\) with \(n + 1\)'s and \(n - 1\)'s such that every partial sum is \(\geq 0\)
Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n + 2)$-gon

2. Lattice paths from $(0, 0)$ to $(n, n)$ staying above $y = x$

3. Sequences of length $2n$ with $n + 1$’s and $n − 1$’s such that every partial sum is $\geq 0$

4. Ways to multiply $n + 1$ numbers together two at a time.
We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!
We claim that these objects are all counted by the Catalan numbers. So there should be **bijections** between the sets!

**Bijection 1:**

| Triangulations of an \((n+2)\)-gon | Multiplication schemes to multiply \(n+1\) numbers |

**Rule:** Label all but one side of the \((n+2)\)-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.
Catalan Bijections

Bijection 2: multiplication schemes to multiply \( n + 1 \) #'s \( \leftrightarrow \) seqs with \( n + 1 \)'s, \( n - 1 \)'s with positive partial sums

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by \( +1 \)'s and the parentheses by \( -1 \)'s.
Catalan Bijections

Bijection 3: seqs with $n+1$'s, $n-1$'s with positive partial sums $\leftrightarrow$ lattice paths $(0,0)$ to $(n,n)$ above $y=x$

A sequence of $+$'s and $-$'s converts to a sequence of $N$’s and $E$’s, which is a path in the lattice.
The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$
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$$C(x) = 1 + xC(x)^2.$$ 

Example. Triangulations of an $(n+2)$-gon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): $x^0$.
2. Every other triangulation has one side ($x$ contribution) and is a sequence of two other triangulations $C(x)^2$. 

Here, $x$ represents one side of the polygon.
Catalan Numbers

Example. Lattice paths \((0, 0)\) to \((n, n)\) above \(y = x\) above. Here, \(x\) represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: \(x^0\).
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (\(x\) contribution). “Between the V & H steps” and “after the H step” is a sequence of two lattice paths \(C(x)^2\).

Therefore, \(C(x) = 1 + xC(x)^2\).
A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$. 
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Do we take the positive or negative root? Check \( x = 0 \).
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Now extract coefficients to prove the formula for $c_n$.

Claim: $\sqrt{1 - 4x} = 1 + \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$. (Next slide.)
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(Next slide.)

Conclusion: \( \frac{1}{2x} (1 - \sqrt{1 - 4x}) = -\frac{1}{2x} \sum_{k \geq 1} \frac{2}{k} \binom{2(k-1)}{k-1} x^k \)
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= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n
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$= \sum_{k \geq 1} \frac{1}{k} \binom{2(k-1)}{k-1} x^{k-1}$

$= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$

Therefore, $c_n = \frac{1}{n+1} \binom{2n}{n}$. 
Expansion of $\sqrt{1 - 4x}$

What is the power series expansion of $\sqrt{1 - 4x}$?

$$\sqrt{1 - 4x} = ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k$$

Expand $\binom{1/2}{k}$
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$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} (-4x)^k$$

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Denom. of $\frac{1}{2}$
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Expand \( \binom{1/2}{k} \)

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2} - k + 1\right)}{k!} (-4x)^k$$

Denom. of \( \frac{1}{2} \)

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!} (-1)^k 4^k x^k$$

Factor \(-2\)'s
Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = (\sqrt{-4x} + 1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k$$

Expand $\binom{1/2}{k}$

$$= 1 + \sum_{k=1}^{\infty} \frac{1/2(1/2-1)\cdots(1/2-k+1)}{k!} (-4x)^k$$

Denom. of $\frac{1}{2}$

$$= 1 + \sum_{k=1}^{\infty} \frac{1/2(-1/2)\cdots(-2k-3)}{k!} (-1)^k 4^k x^k$$

Factor $-2$'s

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k! 2^k} (-1)^k 4^k x^k$$

Simplify; rewrite prod.
Expansion of $\sqrt{1 - 4x}$

What is the power series expansion of $\sqrt{1 - 4x}$?

$$\sqrt{1 - 4x} = ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)\binom{1/2}{k}(-4x)^k$$

Expand $\left(\frac{1}{2}\right)\binom{1/2}{k}$

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}(-4x)^k$$

Denom. of $\frac{1}{2}$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k4^kx^k$$

Factor $-2$’s

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k4^kx^k$$

Simplify; rewrite prod.

$$= 1 + \sum_{k=1}^{\infty} \frac{1\cdot2\cdot3\cdot4\cdots(2k-3)(2k-2)}{k!\cdot2\cdot4\cdots(2k-2)}2^kx^k$$

Write as factorials
Expansion of $\sqrt{1 - 4x}$

What is the power series expansion of $\sqrt{1 - 4x}$?

$$\sqrt{1 - 4x} = ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \left(\frac{1/2}{k!}\right)(-4x)^k$$

Expanding $\left(\frac{1/2}{k!}\right)$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - k + 1)}{k!}(-4x)^k$$

Denominator of $\frac{1}{2}$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k - 3}{2})}{k!}(-1)^k 4^k x^k$$

Factoring $-2$'s

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k - 3)}{k! 2^k}(-1)^k 4^k x^k$$

Simplifying and rewriting the product

$$= 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k - 3) \cdot (2k - 2)}{k! \cdot 2 \cdot 4 \cdots (2k - 2)} 2^k x^k$$

Writing as factorials

$$= 1 + \sum_{k=1}^{\infty} \frac{(2k - 2)!}{k!(2k - 1)! 1 \cdot 2 \cdots (k - 1)} 2^k x^k$$
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Expand $\left(\frac{1}{2}\right)_k$

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Write as factorials

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Expand $\left(\frac{1}{2}\right)$

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$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (1) \cdots (2k-3)}{k! 2^k} (-1)^k 4^k x^k$

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