Catalan Numbers

\[
\begin{array}{cccccccccc}
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \\
1 & 1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 & 16796
\end{array}
\]


\[
c_n = \frac{1}{n+1} \binom{2n}{n}.
\]

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), ... (a^8) to (y^8).

- triangulations of an \((n+2)\)-gon
- lattice paths from \((0, 0)\) to \((n, n)\) above \(y = x\)
- sequences with \(n + 1\)'s, \(n - 1\)'s with positive partial sums
- multiplication schemes to multiply \(n + 1\) numbers
When \( n = 3 \), there are \( c_3 = 5 \) members of these families of objects:

1. Triangulations of an \((n + 2)\)-gon

2. Lattice paths from \((0, 0)\) to \((n, n)\) staying above \( y = x \)

3. Sequences of length \( 2n \) with \( n + 1 \)'s and \( n - 1 \)'s such that every partial sum is \( \geq 0 \)

4. Ways to multiply \( n + 1 \) numbers together two at a time.
Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Bijection 1:

| triangulations of an \((n+2)\)-gon | ↔ | multiplication schemes to multiply \(n + 1\) numbers |

Rule: Label all but one side of the \((n + 2)\)-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.
Catalan Bijections

Bijection 2: multiplication schemes to multiply $n+1$ #s $\leftrightarrow$ seqs with $n$ +1’s, $n$ −1’s with positive partial sums

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by +1’s and the parentheses by −1’s.
Catalan Bijections

Bijection 3: seqs with \( n +1 \)'s, \( n -1 \)'s with positive partial sums \( \leftrightarrow \) lattice paths \((0, 0)\) to \((n, n)\) above \( y = x \)

A sequence of +’s and −’s converts to a sequence of \( N \)'s and \( E \)'s, which is a path in the lattice.
Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$  

Example. 

<table>
<thead>
<tr>
<th>Triangulations of an $(n+2)$-gon</th>
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Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): $x^0$.
2. Every other triangulation has one side ($x$ contribution) and is a sequence of two other triangulations $C(x)^2$. 

Here, $x$ represents one side of the polygon.
Catalan Numbers

Example. Lattice paths (0, 0) to (n, n) above y = x

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: \( x^0 \).
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (\( x \) contribution). “Between the \( V \) & \( H \) steps” and “after the \( H \) step” is a sequence of two lattice paths \( C(x)^2 \).

Therefore, \( C(x) = 1 + xC(x)^2 \).
A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$. Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for $c_n$.

Claim: $\sqrt{1 - 4x} = 1 + \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$. (Next slide.)

Conclusion. $\frac{1}{2x} (1 - \sqrt{1 - 4x}) = -\frac{1}{2x} \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$

$$= \sum_{k \geq 1} \frac{1}{k} \binom{2(k-1)}{k-1} x^{k-1}$$

$$= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$$

Therefore, $c_n = \frac{1}{n+1} \binom{2n}{n}$. 
Expansion of $\sqrt{1 - 4x}$

What is the power series expansion of $\sqrt{1 - 4x}$?

$$\sqrt{1 - 4x} = ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right) (-4x)^k$$

Expand $\left(\frac{1}{k}\right)$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-k+1\right) \frac{(-4x)^k}{k!}$$

Denom. of $\frac{1}{k}$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{-1}{2}\right) \cdots \left(\frac{-2k-3}{2}\right) \frac{(-1)^k 4^k x^k}{k!}$$

Factor $-2$’s

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1) \cdots (2k-3)}{k! 2^k} (-1)^k 4^k x^k$$

Simplify; rewrite prod.

$$= 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-3)(2k-2)}{k! \cdot 2 \cdot 4 \cdots (2k-2)} 2^k x^k$$

Write as factorials

$$= 1 + \sum_{k=1}^{\infty} \frac{(2k-2)!}{k! (2^{k-1}) 1 \cdot 2 \cdots (k-1)} 2^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k} \frac{(2k-2)!}{(k-1)! (k-1)!} x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k} \left(\frac{2(k-1)}{k-1}\right) x^k$$