Example: Fruit baskets

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Example. In how many ways we can create a fruit basket with $n$ pieces of fruit, where we have an infinite supply of apples and bananas, with the added constraints:

- The number of apples is even.
- The number of bananas is a multiple of five.
- The number of oranges is at most four.
- The number of pears is zero or one.
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- The number of apples is even.
- The number of bananas is a multiple of five.
- The number of oranges is at most four.
- The number of pears is zero or one.

Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of $x^k$. 
Example: Rolling dice

Example. When two standard six-sided dice are rolled, what is the distribution of the sums that appear?

Solution. The generating function for one die is $D(x) =$
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Solution. Find two generating functions $F(x)$ and $G(x)$ such that $F(x)G(x) = D^2(x)$ and $F(1) = G(1) = 6$. Rearrange the factors: $D(x)^2 = x^2(1 + x)^2(1 − x + x^2)^2(1 + x + x^2)^2$. 
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\]

Rearrange the factors:

\[
D(x)^2 = x^2 (1 + x)^2 (1 - x + x^2)^2 (1 + x + x^2)^2.
\]

\[
= [x(1 + x)(1 + x + x^2)] \cdot [x(1 - x + x^2)^2(1 + x)(1 + x + x^2)]
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\[
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\]

Die \( F: \{1, 2, 2, 3, 3, 4\} \) and die \( G: \{1, 3, 4, 5, 6, 8\} \)
Example. Determine a formula for the entries of the sequence \( \{a_k\}_{k \geq 0} \) that satisfies \( a_0 = 0 \) and the recurrence \( a_{k+1} = 2a_k + 1 \) for \( k \geq 0 \).
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Solution. Use generating functions: define \( A(x) = \sum_{k \geq 0} a_k x^k \).

Step 1: Multiply both sides of the recurrence by \( x^{k+1} \) and sum over all \( k \):

\[
\sum_{k \geq 0} a_{k+1} x^{k+1} = \sum_{k \geq 0} (2a_k + 1) x^{k+1}
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Solving recurrence relations

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Step 2: Massage the sums to find copies of \( A(x) \).
LHS: Re-index, find missing term; RHS: separate into pieces.

\[
\sum_{k \geq 1} a_k x^k = \sum_{k \geq 0} 2a_k x^{k+1} + \sum_{k \geq 0} x^{k+1}
\]

Conversion to functions of \( A(x) \):
Step 3: Solve for the compact form of $A(x)$.

$$A(x) = \frac{x}{(1 - 2x)(1 - x)}$$
Solving recurrence relations

**Step 3:** Solve for the compact form of \( A(x) \).

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A(x) = \frac{x}{(1 - 2x)(1 - x)}
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**Step 4:** Extract the coefficients.

When the degree of the numerator is smaller than the degree of the denominator, we can use partial fractions to determine an expression for \( A(x) \) of the form:

\[
A(x) = \frac{C_1}{1 - 2x} + \frac{C_2}{1 - x}
\]

Solving gives \( A(x) = \frac{1}{1 - 2x} + \frac{-1}{1 - x} \); each of which can be expanded:
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\[
A(x) = \sum_{k \geq 0} 2^k x^k + \sum_{k \geq 0} (-1) x^k = \sum_{k \geq 0} (2^k - 1) x^k
\]

Therefore, \( a_k = 2^k - 1 \).
A closed form for Fibonacci numbers

Example. Solve the recurrence relation $f_{k+2} = f_{k+1} + f_k$ with initial conditions $f_0 = 0$ and $f_1 = 1$. 
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$$\sum_{k \geq 2} f_k x^k = x \sum_{k \geq 1} f_k x^k + x^2 \sum_{k \geq 0} f_k x^k$$

Therefore, $F(x) - x - 0 = x(F(x) - 0) + x^2 F(x)$, so

$$F(x) = \frac{x}{1 - x - x^2}$$
A closed form for Fibonacci numbers

So the Fibonacci numbers have generating function $x/(1 - x - x^2)$. The roots of $(1 - x - x^2) = (1 - r_+x)(1 - r_-x)$ are $r_\pm = (1 \pm \sqrt{5})/2$. 
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F(x) = \frac{1}{\sqrt{5}} \frac{1}{1 - r_+x} - \frac{1}{\sqrt{5}} \frac{1}{1 - r_-x}
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Therefore,

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\sum_{k \geq 0} f_k x^k = \sum_{k \geq 0} \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k x^k - \sum_{k \geq 0} \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k x^k
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and we conclude that

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and we conclude that $f_k = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k$.

As $k \rightarrow +\infty$, the second term goes to zero, so $f_k \approx \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k$. 


A closed form for Fibonacci numbers

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Therefore, \( \sum_{k \geq 0} f_k x^k = \sum_{k \geq 0} \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k x^k - \sum_{k \geq 0} \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k x^k \)

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Practicality: \((1 + \sqrt{5})/2 \approx 1.61803\) and 1 mi \( \approx 1.609344 \) km
Solving recurrence relations with repeated roots

With repeated roots in the denominator, the result is not quite as nice.

**Example.** Find the partial fraction decomposition of \( \frac{x}{(1-2x)^2(1+5x)} \).

Since \((1 - 2x)^2 \) is a repeated root,

\[
\frac{x}{(1-2x)^2(1+5x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x}.
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\]

Clearing the denominator gives:

\[
x = A(1 - 2x)(1 + 5x) + B(1 + 5x) + C(1 - 2x)^2.
\]

When \( x = \frac{1}{2}, \) \( \frac{1}{2} = 0 + B(1 + \frac{5}{2}) + 0; \) so \( B = \frac{1}{7}. \)

When \( x = -\frac{1}{5}, \) \( -\frac{1}{5} = 0 + 0 + C(1 + \frac{2}{5})^2; \) so \( C = \frac{-5}{49}. \)
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Equating the coefficients of \(x^0\), we see \(A + B + C = 0\). We conclude \(A = \frac{-2}{49}\).
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\[
\frac{x}{(1 - 2x)^2(1 + 5x)} = \frac{-2}{49} + \frac{7}{49} + \frac{-5}{49}.
\]
Example. Let \( \{h_k\}_{k \geq 0} \) be a sequence satisfying

\[
h_k + h_{k-1} - 16h_{k-2} + 20h_{k-3} = 0,
\]

with initial conditions \( h_0 = 1 \), \( h_1 = 1 \), and \( h_2 = -1 \).
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Find the generating function and formula for \( h_k. \)

\[
h(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_kx^k + \cdots,
\]

\[
+ xh(x) = h_0x + h_1x^2 + h_2x^3 + \cdots + h_{k-1}x^k + \cdots,
\]

\[
- 16x^2 h(x) = -16h_0x^2 - 16h_1x^3 + \cdots - 16h_{k-2}x^k + \cdots,
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\[
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\[
\begin{align*}
h(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_k x^k + \cdots, \\
+xh(x) &= h_0 x + h_1 x^2 + h_2 x^3 + \cdots + h_{k-1} x^k + \cdots, \\
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Therefore, \( h(x) = \)

Since \( \frac{1}{(1-y)^m} = \sum_{k \geq 0} \binom{m}{k} y^k \),
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Therefore, \( h(x) = \)

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we see \( \frac{1}{(1-2x)^2} = \sum \binom{k+1}{k}(2x)^k = \sum (k+1)2^kx^k. \)

We conclude that \( h_k = \)