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Solution.
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Solution.

Let $S$ be the set of students who play soccer and $B$ be the set of students who play basketball. Then, $|S \cup B| = |S| + |B|$. 

\[
\begin{array}{c}
\text{Set } S \\
\cap \\
\text{Set } B
\end{array}
\]
Principle of Inclusion-Exclusion

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) (\( \mathcal{U} \) for universe) and the sets \( A_i \) are pairwise disjoint, we have \( |A| = |A_1| + \cdots + |A_k| \).
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When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ and the $A_i$ are not pairwise disjoint, we must apply the principle of inclusion-exclusion to determine $|A|$:
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$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
Principle of Inclusion-Exclusion

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) (\( \mathcal{U} \) for universe) and the sets \( A_i \) are \textit{pairwise disjoint}, we have \( |A| = |A_1| + \cdots + |A_k| \).

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) and the \( A_i \) are \textbf{not} pairwise disjoint, we must apply the \textit{principle of inclusion-exclusion} to determine \( |A| \):

\[
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Principle of Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ ($\mathcal{U}$ for universe) and the sets $A_i$ are 
pairwise disjoint, we have $|A| = |A_1| + \cdots + |A_k|$.

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ and the $A_i$ are not pairwise disjoint, we must apply the principle of inclusion-exclusion to determine $|A|:

\begin{align*}
|A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\
|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\
&\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\
|A_1 \cup \cdots \cup A_m| &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots
\end{align*}
Principle of Inclusion-Exclusion

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) (\( \mathcal{U} \) for universe) and the sets \( A_i \) are *pairwise disjoint*, we have \(|A| = |A_1| + \cdots + |A_k|\).

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) and the \( A_i \) are *not* pairwise disjoint, we must apply the principle of inclusion-exclusion to determine \(|A|\):

\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|
\]

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|
\]

\[
- |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\]

\[
|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots
\]

It may be more convenient to apply inclusion/exclusion where the \( A_i \) are *forbidden* subsets of \( \mathcal{U} \), in which case ________________.
mmm. ...PIE

The key to using the principle of inclusion-exclusion is determining the right choice of $A_i$. The $A_i$ and their intersections should be easy to count and easy to characterize.
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**Notation:** $\pi = p_1p_2\cdots p_n$ is the one-line notation for a permutation of $[n]$ whose first element is $p_1$, second element is $p_2$, etc.

**Example.** How many permutations $\pi = p_1p_2\cdots p_n$ are there in which at least one of $p_1$ and $p_2$ are even?
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**Solution.** Let $U$ be the set of $n$-permutations. Let $A_1$ be the set of permutations where $p_1$ is even. Let $A_2$ be the set of permutations where $p_2$ is even.

In words, $A_1 \cap A_2$ is the set of $n$-permutations _________________
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**Applying PIE:** So $|A_1 \cup A_2| =$
Example. Find the number of integers between 1 and 1000 that are not divisible by 5, 6, or 8.
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Solution. Let $\mathcal{U} = \{ n \in \mathbb{Z} \text{ such that } 1 \leq n \leq 1000 \}$.
Let $A_1 \subset \mathcal{U}$ be the multiples of 5, $A_2 \subset \mathcal{U}$ be the multiples of 6, and $A_3 \subset \mathcal{U}$ be the multiples of 8. We want $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3|$. 

### PIE

mmm...
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And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$
Combinations with Repetitions

Quick review

1. How many ways are there to choose $k$ elements out of the set
   \[\{1 \cdot a_1, 1 \cdot a_2, \cdots, 1 \cdot a_n\}\]?
Combinations with Repetitions

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2. How many ways are there to choose \( k \) elements out of the set \( \{k \cdot a_1, k \cdot a_2, \ldots, k \cdot a_n\} \) (really \( \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\} \)?)
Combinations with Repetitions

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What we would like to calculate is:

In how many ways can we choose \( k \) elements out of an arbitrary multiset?

Now, it’s as easy as PIE.
**Example.** Determine the number of 10-combinations of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.
Combinations with Repetitions

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Game plan: Let $U$ be the set of 10-combs of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Use PIE to remove the 10-combs that violate the conditions of $S$. 
Example. Determine the number of 10-combinations of the multiset \( S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\} \).

Game plan: Let \( \mathcal{U} \) be the set of 10-combs of \( \{\infty \cdot a, \infty \cdot b, \infty \cdot c\} \). Use PIE to remove the 10-combs that violate the conditions of \( S \). Define \( A_1 \) to be 10-combs that include at least ___ a’s.
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Define $A_1$ to be 10-combs that include at least ___ $a$’s.
Define $A_2$ to be 10-combs that include at least ___ $b$’s.
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In words, $A_1 \cap A_2$ are those 10-combs that
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Now calculate: $|\mathcal{U}| = |A_1| =$
Combinations with Repetitions

Example. Determine the number of 10-combinations of the multiset \( S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\} \).

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In words, \( A_1 \cap A_2 \) are those 10-combs that
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\( A_1 \cap A_2 \cap A_3 \)

Now calculate: \( |U| = |A_1| = |A_2| = \binom{3}{5} \)
\( |A_3| = \binom{3}{4} \)
\( |A_1 \cap A_2| = 3 \)
\( |A_1 \cap A_3| = 1 \)
\( |A_2 \cap A_3| = 0 \)
\( |A_1 \cap A_2 \cap A_3| = 0 \)

And finally: So \( |U| - |A_1 \cup A_2 \cup A_3| = \)
At a party, 10 partygoers check their hats. They “have a good time”, and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his/her own hat?
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This is a derangement of ten objects.

**Definition:** An $n$-derangement is an $n$-permutation $\pi = p_1p_2\cdots p_n$ such that $p_1 \neq 1$, $p_2 \neq 2$, $\cdots$, $p_n \neq n$.

**Note:** A derangement is a permutation without fixed points $\pi(i) \neq i$. 
Derangements

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**Note:** A derangement is a permutation without fixed points \( \pi(i) = i \).

**Notation:** We let \( D_n \) be the number of all \( n \)-derangements.

When you see \( D_n \), think combinatorially: “The number of ways to return \( n \) hats to \( n \) people so no one gets his/her own hat back”
Calculating the number of derangements

Example. Calculate $D_n$.

Solution. Let $\mathcal{U}$ be the set of all $n$-permutations.
Remove bad permutations using PIE.
For all $i$ from 1 to $n$, define $A_i$ to be $n$-perms where $p_i = i$. 
Calculating the number of derangements

Example. Calculate $D_n$.

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In general, $A_{i_1} \cap \cdots \cap A_{i_k}$ are $n$-perms with $p_{i_1} = i_1$, $\cdots$, $p_{i_k} = i_k$.

**Now calculate**: $|\mathcal{U}| = \quad |A_1| = \quad |A_2| = \quad$
Calculating the number of derangements

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Now calculate: $|\mathcal{U}| = \quad |A_1| = \quad |A_2| = \quad$

For all $i$ and $j$, $|A_i \cap A_j| = \quad$
Calculating the number of derangements

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Now calculate: $|\mathcal{U}| = A_1 = A_2 = \cdots$ 

For all $i$ and $j$, $|A_i \cap A_j| = \cdots$

When intersecting $k$ sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| = \cdots$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots$
Calculating the number of derangements

Example. Calculate $D_n$.

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For all $i$ and $j$, $|A_i \cap A_j| =$

When intersecting $k$ sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots$

Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \cdots \cup A_n| =$
Randomly returning hats

Upon simplification, we see

\[ D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \]

\[ = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \]

\[ = n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \]
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Recall: Taylor series expansion of \( e^x \):
\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \]
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\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \]

Plug in \( x = -1 \) and truncate after \( n \) terms to see that

\[ e^{-1} \approx \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \]
Randomly returning hats

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\]

Conclusion: If \(n\) people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is \(D_n/n!\), which is approximately \(1/e \approx 37\%\).
Recall: The combinatorial interpretation of $D_n$ is: “The number of ways to return $n$ hats to $n$ people so no one gets his/her own hat back”

Example. Prove the following recurrence relation for $D_n$ combinatorially.

$$D_n = (n - 1)(D_{n-2} + D_{n-1})$$
A formula for Stirling numbers (p. 90)

Recall: \( S(n, k) = \{n\choose k} \) is the number of partitions of the set \([n]\) into exactly \(k\) parts
A formula for Stirling numbers (p. 90)

Recall: $S(n, k) = \left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the number of partitions of the set $[n]$ into exactly $k$ parts.
A formula for Stirling numbers (p. 90)

Recall: \( S(n, k) = \binom{n}{k} \) is the number of partitions of the set \([n]\) into exactly \(k\) parts, and \(k!S(n, k)\) is the number of \textit{onto functions} \([n] \to [k]\).
Recall: \( S(n, k) = \{^n_k\} \) is the number of partitions of the set \([n]\) into exactly \(k\) parts, and \(k!S(n, k)\) is the number of \textbf{onto functions} \([n]\rightarrow[k]\).

**Question:** What is a formula for \(S(n, k)\)?
A formula for Stirling numbers (p. 90)

Recall: \( S(n, k) = \binom{n}{k} \) is the number of partitions of the set \([n]\) into exactly \(k\) parts, and \(k!S(n, k)\) is the number of onto functions \([n]\to[k]\).

**Question:** What is a formula for \(S(n, k)\)?

**Solution.** We will find the number of surjections from \([n]\) to \([k]\).

Use PIE with \(\mathcal{U} = \text{set of all functions from } [n] \to [k]\).

We will remove the “bad” functions where the range is not \([k]\).
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**Question:** What is a formula for $S(n, k)$?

**Solution.** We will find the number of surjections from $[n]$ to $[k]$. Use PIE with $\mathcal{U} =$ set of all functions from $[n]$ to $[k]$. We will remove the “bad” functions where the range is not $[k]$. Define $A_i$ be the set of functions $f : [n] \rightarrow [k]$ where $i$ is not “hit”.

(Careful: change of notation!!)
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In words, $A_{i_1} \cap \cdots \cap A_{i_j}$ are functions where none of $i_1$ through $i_j$ are elements of the image.
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**We calculate:** $|\mathcal{U}| = k^n$, $|A_i| = (k - 1)^n$, $|A_i \cap A_j| = (k - 2)^n$ When intersecting $j$ sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$. 
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When intersecting $j$ sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$.

Therefore, $k!S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n$; we conclude $S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n$.
A formula for Stirling numbers (p. 90)

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Define $A_i$ be the set of functions $f : [n] \rightarrow [k]$ where $i$ is not “hit”.

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**We calculate:** $|\mathcal{U}| = k^n$, $|A_i| = (k - 1)^n$, $|A_i \cap A_j| = (k - 2)^n$

When intersecting $j$ sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$.

Therefore, $k!S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n$; we conclude

$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$. 
A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \binom{n}{k} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n$$
A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \left\{ \binom{n}{k} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$
A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \left\{ \begin{array}{c} n \\ k \end{array} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

$$= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n$$
A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \binom{n}{k} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

$$= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} j^n}{(k-j)!j!}$$
A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

\[ B_n = \sum_{k \geq 0} \left\{ \binom{n}{k} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n \]

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\]

\[
= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} j^n}{(k-j)! j!}
\]

\[
= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{k \geq j} \frac{(-1)^{k-j}}{(k-j)!}
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A formula for Bell numbers (p. 166)

Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

$$= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} j^n}{(k-j)! \ j!}$$

$$= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! \ j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!}$$
A formula for Bell numbers (p. 166)

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$$B_n = \sum_{k \geq 0} \binom{n}{k} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

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$$= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! \ j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!} \frac{1}{e} = \sum_{j \geq 0} \frac{j^n}{j!} \frac{1}{e}$$
Recall: $B_n$ is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \binom{n}{k} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

$$= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} j^n}{(k-j)! \ j!}$$

$$= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! \ j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!} = \sum_{j \geq 0} \frac{j^n}{j! \ e}$$

Theorem 4.3.3. For any $n > 0$, $B_n = \frac{1}{e} \sum_{j \geq 0} \frac{j^n}{j!}$. 
A formula for Bell numbers (p. 166)

Recall: \( B_n \) is the number of partitions of \([n]\) into any number of parts.
Manipulate our expression from prev. page to find a formula.

\[
B_n = \sum_{k \geq 0} \binom{n}{k} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n
\]

\[
= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} j^n}{(k-j)!} j!
\]

\[
= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)!} j = \sum_{j \geq 0} j^n \sum_{m \geq 0} \frac{(-1)^m}{m!} = \sum_{j \geq 0} j^n \frac{1}{e}
\]

**Theorem 4.3.3.** For any \( n > 0 \), \( B_n = \frac{1}{e} \sum_{j \geq 0} \frac{j^n}{j!} \).

For example, \( B_5 = \frac{1}{e} \left( \frac{0^5}{0!} + \frac{1^5}{1!} + \frac{2^5}{2!} + \frac{3^5}{3!} + \frac{4^5}{4!} + \frac{5^5}{5!} + \cdots \right) = 52. \)