Introduction to Symmetry

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In order to approach counting questions involving symmetry rigorously, we use the mathematical notion of *equivalence relation*. 
**Equivalence relations**

*Definition:* An equivalence relation $\mathcal{E}$ on a set $A$ satisfies the following properties:

- **Reflexive:** For all $a \in A$, $a \mathcal{E} a$.
- **Symmetric:** For all $a, b \in A$, if $a \mathcal{E} b$, then $b \mathcal{E} a$.
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- **Reflexive:** For all $a \in A$, $aEa$.
- **Symmetric:** For all $a, b \in A$, if $aEb$, then $bFa$.
- **Transitive:** For all $a, b, c \in A$, if $aEb$, and $bEc$, then $aEc$.

Example. When sitting four people at a round table, let $A$ be all 4-permutations. We say $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ are “equivalent” ($aEb$) if they are rotations of each other.

Verify that $E$ is an equivalence relation.
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Equivalence classes

It is natural to investigate the set of all elements related to $a$:

**Definition:** The **equivalence class containing** $a$ is the set

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<tr>
<th>Class</th>
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<td><strong>1</strong></td>
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Our original question asks to count *equivalence classes* (!).
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Counting using symmetry— §1.4

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- Our original question asks to count equivalence classes (!).
- **Theorem 1.4.3.** If \( a \mathcal{E} b \), then \( \mathcal{E}(a) = \mathcal{E}(b) \).
- Every element of \( A \) is in *one* and *only one* equivalence class.
  - We say: “The equivalence classes of \( \mathcal{E} \) partition \( A \).”
Equivalence classes partition $A$

*Definition:* A **partition** of a set $S$ is a set of non-empty disjoint subsets of $S$ whose union is $S$.

**Example.** Partitions of $S = \{\ast, \heartsuit, \clubsuit, ?\}$ include:

- $\left\{\left\{\ast, \heartsuit\right\}, \{?\}, \{\clubsuit\}\right\}$
- $\left\{\left\{\heartsuit, \clubsuit\right\}, \{\ast, ?\}\right\}$

Every element is in some subset and no element is in multiple subsets.
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**Key idea:** (Thm 1.4.5) The set of equivalence classes of $A$ partitions $A$.

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- Every equivalence class is non-empty.
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The equivalence principle: (p. 37) Let $E$ be an equivalence relation on a finite set $A$. If every equivalence class has size $C$, then $E$ has $|A|/C$ equivalence classes.
Permutations of multisets

Example. How many different orderings are there of the letters in the word MISSISSIPPI?

**Setup:** If the letters were all distinguishable, we would have a permutation of 11 letters, \( \{M, P, P, I, I, I, I, S, S, S, S\} \).
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Alternatively, count directly.

- In how many ways can you position the \( S \)'s?
- With \( S \)'s placed, how many choices for the \( I \)'s?
- With \( S \)'s, \( I \)'s placed, how many choices for the \( P \)'s?
- With \( S \)'s, \( I \)'s, \( P \)'s placed, how many choices for the \( M \)?
The Equivalence Principle (Group Activity)

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Setup: Let $A$ be the set of 10-lists, $(a_1, a_2, \ldots, a_9, a_{10}) = a \in A$. The list $a$ will represent the pairings $\{\{a_1, a_2\}, \ldots, \{a_9, a_{10}\}\}$. 
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Define two lists $a$ and $b$ to be equivalent if they give the same pairings. [For example, $(3, 2, 9, 10, 1, 5, 8, 7, 4, 6) \equiv (2, 3, 9, 10, 1, 5, 6, 4, 8, 7)$.] (Why is this an equivalence relation?)
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*For example, $(3, 2, 9, 10, 1, 5, 8, 7, 4, 6) \equiv (2, 3, 9, 10, 1, 5, 6, 4, 8, 7).*

(Why is this an equivalence relation?)

We ask: How many different 10-lists are in the same equivalence class?

**Answer:**

By the equivalence principle,
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Example. Let $A$ be the subsets of $[4]$. Define $S \leq T$ when $|S| = |T|$. Determine the number of conjugacy classes of $\mathcal{E}$. 
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Solution. (NOT) We know that $E(\{1\}) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, of size 4. Since $|A| = 24$, there are $\frac{24}{4} = 6$ conjugacy classes.
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Solution. The conjugacy classes correspond to ________________.