Counting integral solutions

**Question:** How many non-negative integer solutions are there of \( x_1 + x_2 + x_3 + x_4 = 10 \)?

- Give some examples of solutions.
- Characterize what solutions look like.
- A combinatorial object with a similar flavor is:

In general, the number of non-negative integer solutions to \( x_1 + x_2 + \cdots + x_n = k \) is _______.

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- A combinatorial object with a similar flavor is:

In general, the number of non-negative integer solutions to \( x_1 + x_2 + \cdots + x_n = k \) is _______.

**Question:** How many positive integer solutions are there of \( x_1 + x_2 + x_3 + x_4 = 10 \), where \( x_4 \geq 3 \)?
The sum principle

Often it makes sense to break down your counting problem into smaller, disjoint, and easier-to-count sub-problems.

Example. How many integers from 1 to 999999 are palindromes?
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Answer: Condition on how many digits.

- Length 1:
- Length 2:
- Length 3:
- Length 4:
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- Total:
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This illustrates the sum principle:

Suppose the objects to be counted can be broken into $k$ disjoint and exhaustive cases. If there are $n_j$ objects in case $j$, then there are $n_1 + n_2 + \cdots + n_k$ objects in all.
Counting pitfalls

When counting, there are two common pitfalls:
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  ▶ **Ask:** Did I miss something?

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- **Undercounting**
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  - **Ask:** Did I miss something?

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  - Often, misapplying the product principle.
  - **Ask:** Do cases need to be counted in different ways?
  - **Ask:** Does the same object appear in multiple ways?
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  - **Ask:** Does the same object appear in multiple ways?

**Common example:** A deck of cards.

There are four suits: Diamond ♦️, Heart ♥️, Club ♣️, Spade ♠️.
Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.
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When counting, there are two common pitfalls:

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  - Often, *forgetting cases* when applying the sum principle.
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Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

**Example.** Suppose you are dealt two diamonds between 2 and 10.
In how many ways can the product be even?
Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a Heart 🖤?
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Remember to ask: Do cases need to be counted in different ways?
Example. How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555  Non-examples: 1231, 9898.
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Counting the complement

Q1: How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

—Compare this to—

Q2: How many 4-lists taken from [9] have no pairs of adjacent elements equal?

What can we say about:

Q1: Q2:
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Game plan:
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  - Apply the multiplication principle. Total:
- Divide to find the probability.

$$\frac{3744}{2598960} \approx 0.14\%$$
Introduction to Bijections

**Key tool:** A useful method of proving that two sets $A$ and $B$ are of the same size is by way of a *bijection*.

A **bijection** is a function or rule that pairs up elements of $A$ and $B$. 
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**Example.** The set $A$ of subsets of $\{s_1, s_2, s_3\}$ are in bijection with the set $B$ of binary words of length 3.

Set $A$: \[
\emptyset, \ \{s_1\}, \ \{s_2\}, \ \{s_1, s_2\}, \ \{s_3\}, \ \{s_1, s_3\}, \ \{s_2, s_3\}, \ \{s_1, s_2, s_3\}\]

Set $B$: \[
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Bijection: \( \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \)

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**Rule:** Given $a \in A$, ($a$ is a subset), define $b \in B$ ($b$ is a word):
- If $s_i \in a$, then letter $i$ in $b$ is 1.
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**Difficulties:**
- Finding the function or rule (requires rearranging, ordering)
- Proving the function or rule (show it IS a bijection).
What is a Function?

Reminder: A function $f$ from $A$ to $B$ (write $f : A \rightarrow B$) is a rule where for each element $a \in A$, $f(a)$ is defined as an element $b \in B$ (write $f : a \mapsto b$).
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$$\text{rng}(f) = \{ b \in B : f(a) = b \text{ for at least one } a \in A \}$$
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Example. Let $A$ be the set of 3-subsets of $[n]$ and let $B$ be the set of 3-lists of $[n]$. Then define $f : A \rightarrow B$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in A$ (with $i_1 \leq i_2 \leq i_3$) to the word $i_1i_2i_3 \in B$.

**Question:** Is $\text{rng}(f) = B$?
What is a Bijection?

Definition: A function $f : A \rightarrow B$ is **one-to-one** (an injection) when
For each $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. 
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Definition: A function $f : A \rightarrow B$ is onto (a surjection) when For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$. “Every output gets hit.”
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The function from the previous page is ________________.
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What is an example of a function that is onto and not one-to-one?
Example. Use a bijection to prove that \( \binom{n}{k} = \binom{n}{n-k} \) for \( 0 \leq k \leq n \).
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Proof. Let \( A \) be the set of \( k \)-subsets of \([n]\) and let \( B \) be the set of \((n-k)\)-subsets of \([n]\).

A bijection between \( A \) and \( B \) will prove \( \binom{n}{k} = |A| = |B| = \binom{n}{n-k} \).
Proving a Bijection

Example. Use a bijection to prove that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$.

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A bijection between $A$ and $B$ will prove $\binom{n}{k} = |A| = |B| = \binom{n}{n-k}$.

Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

$$\left\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \right\} \leftrightarrow \left\{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\} \right\}$$
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\begin{align*}
\{1, 2\}, \{1, 3\} & \leftrightarrow \{1, 2, 3\}, \{1, 2, 4\} \\
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\{3, 5\}, \{4, 5\} & \leftrightarrow \{2, 4, 5\}, \{3, 4, 5\}
\end{align*}
\]

Guess: Let $S$ be a $k$-subset of $[n]$. Perhaps $f(S) = \underline{\hspace{2cm}}$. 


Step 2: Prove $f$ is well defined.

The function $f$ is well defined. If $S$ is any $k$-subset of $[n]$, then $S^c$ is a subset of $[n]$ with $n - k$ members. Therefore $f : A \rightarrow B$. 

Proving a Bijection
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**Step 3: Prove $f$ is a bijection.**
**Strategy.** Prove that $f$ is both one-to-one and onto.
Proving a Bijection

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**Step 3: Prove $f$ is a bijection.**

**Strategy.** Prove that $f$ is both one-to-one and onto.

**$f$ is 1-to-1:** Suppose that $S_1$ and $S_2$ are two $k$-subsets of $[n]$ such that $f(S_1) = f(S_2)$. That is, $S_1^c = S_2^c$. This means that for all $i \in [n]$, then $i \notin S_1$ if and only if $i \notin S_2$. Therefore $S_1 = S_2$ and $f$ is 1-to-1.
**Proving a Bijection**

**Step 2: Prove \( f \) is well defined.**

The function \( f \) is well defined. If \( S \) is any \( k \)-subset of \([n]\), then \( S^c \) is a subset of \([n]\) with \( n - k \) members. Therefore \( f : A \rightarrow B \).

**Step 3: Prove \( f \) is a bijection.**

Strategy. Prove that \( f \) is both one-to-one and onto.

\( f \) is 1-to-1: Suppose that \( S_1 \) and \( S_2 \) are two \( k \)-subsets of \([n]\) such that \( f(S_1) = f(S_2) \). That is, \( S_1^c = S_2^c \). This means that for all \( i \in [n] \), then \( i \notin S_1 \) if and only if \( i \notin S_2 \). Therefore \( S_1 = S_2 \) and \( f \) is 1-to-1.

\( f \) is onto: Suppose that \( T \in B \) is an \((n - k)\)-subset of \([n]\).

We must find a set \( S \in A \) satisfying \( f(S) = T \). Choose \( S = \ldots \). Then \( S \in A \) (why?), and \( f(S) = S^c = T \), so \( f \) is onto.

We conclude that \( f \) is a bijection and therefore, \( \binom{n}{k} = \binom{n}{n-k} \).
Using the Inverse Function

When $f : A \to B$ is 1-to-1, we can define $f$’s inverse. We write $f^{-1}$, and it is a function from $\text{rng}(f)$ to $A$. It is defined via $f$. If $f : a \mapsto b$, then $f^{-1} : b \mapsto a$. 
Using the Inverse Function

When \( f : A \to B \) is 1-to-1, we can define \( f \)'s inverse. We write \( f^{-1} \), and it is a function from \( \text{rng}(f) \) to \( A \). It is defined via \( f \). If \( f : a \mapsto b \), then \( f^{-1} : b \mapsto a \).

**Caution:** When \( f \) is a function from \( A \) to \( B \), \( f^{-1} \) might not be a function from \( B \) to \( A \).
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**Theorem.** Suppose that \( A \) and \( B \) are finite sets and that \( f : A \to B \) is a function. If \( f^{-1} \) is a function with domain \( B \), then \( f \) is a bijection.
Using the Inverse Function

When $f : A \to B$ is 1-to-1, we can define $f$’s inverse. We write $f^{-1}$, and it is a function from $\text{rng}(f)$ to $A$. It is defined via $f$. If $f : a \mapsto b$, then $f^{-1} : b \mapsto a$.

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**Theorem.** Suppose that $A$ and $B$ are finite sets and that $f : A \to B$ is a function. If $f^{-1}$ is a function with domain $B$, then $f$ is a bijection.

**Proof.** Since $f^{-1}$ is only defined when $f$ is 1-to-1, we need only prove that $f$ is onto. Suppose $b \in B$. By assumption, $f^{-1}(b) \in A$ exists and $f(f^{-1}(b)) = b$. So $f$ is onto, and is a bijection.
Using the Inverse Function

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**Consequence:** An alternative method for proving a bijection is:

- Find a rule $g : B \to A$ which always takes $f(a)$ back to $a$.
- Verify that the domain of $g$ is *all of* $B$. 
Using the Inverse Function

**Example.** There exists as many even-sized subsets of \([n]\) as odd-sized subsets of \([n]\).
Using the Inverse Function

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\[
\begin{align*}
\text{even:} & \quad \{ \emptyset, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\} \} \\
\text{odd:} & \quad \{ \{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2, s_3\} \}
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- odd: \(\{\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2, s_3\}\}\)

Proof. Let \(A\) be the set of even-sized subsets of \([n]\) and let \(B\) be the set of odd-sized subsets of \([n]\). Consider the function

\[
f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.
\]

- \(f : A \rightarrow B\) is a well defined function from \(A\) to \(B\) (why?).
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- \(f^{-1}\) exists and equals \(f\) (why?)
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**Example.** There exists as many even-sized subsets of $[n]$ as odd-sized subsets of $[n]$.

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**Proof.** Let $A$ be the set of even-sized subsets of $[n]$ and let $B$ be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

- $f : A \rightarrow B$ is a well defined function from $A$ to $B$ (why?).
- $f^{-1}$ exists and equals $f$ (why?) and has domain $B$ (why?).
Using the Inverse Function

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\(\implies\) \(f^{-1}\) exists and equals \(f\) (why?) and has domain \(B\) (why?). Therefore, \(f\) is a bijection, proving the statement, as desired.
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**Eyebrow-Raising Consequence:** \[\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.\]