Combinatorial statistics

Given a set of combinatorial objects \( \mathcal{A} \), a **combinatorial statistic** is an integer given to every element of the set.

In other words, it is a function \( \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0} \).

**Example.** Let \( \mathcal{S} \) be the set of subsets of \( \{1, 2, 3\} \). The cardinality of a set is a combinatorial statistic on \( \mathcal{S} \).

\[
\begin{align*}
|\emptyset| &= 0 & |\{1\}| &= 1 & |\{2\}| &= 1 & |\{3\}| &= 1 \\
|\{1, 2\}| &= 2 & |\{1, 3\}| &= 2 & |\{2, 3\}| &= 2 & |\{1, 2, 3\}| &= 3
\end{align*}
\]

Combinatorial statistics provide a *refinement* of counting.

\[\begin{array}{c|c|c|c|c}
8 & 0 & 1 & 2 & 3 \\
\hline
1 & 3 & 3 & 1
\end{array}\]

\[\emptyset \quad \{1\} \quad \{2\} \quad \{3\} \quad \{1, 2\} \quad \{1, 3\} \quad \{2, 3\} \quad \{1, 2, 3\}\]
More statistics

Questions involving combinatorial statistics:

- What is the *distribution* of the statistics?
- What is the *average size* of an object in the set?
- Which statistics have the same distribution?
  - Insight into their structure.
  - Provides non-trivial bijections in the set?

A especially rich playground involves *permutation statistics*.

Representations of permutations

One-line notation: $\pi = 4 \ 1 \ 6 \ 2 \ 5 \ 3$  
Cycle notation: $\pi = (1 \ 4 \ 2)(3 \ 6)(5)$

String diagram:

(only two crossings at a time)
**Descent statistic**

*Definition:* Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation. A **descent** is a position $i$ such that $\pi_i > \pi_{i+1}$. Define $\text{des}(\pi)$ to be the number of descents in $\pi$.

**Example.** When $\pi = 416253$, $\text{des}(\pi) = 3$ since $4 \downarrow 1$, $6 \downarrow 2$, $5 \downarrow 3$.

**Question:** How many $n$-permutations have $d$ descents?

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<th>$n \setminus d$</th>
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What are the possible values for $\text{des}(\pi)$?

Note the symmetry. If $\pi$ has $d$ descents, its reverse $\hat{\pi}$ has ____ descents.

These are the **Eulerian numbers**.
Eulerian Numbers

**Definition:** \( A_{n,k} \) = number of \( n \)-permutations with \( k - 1 \) descents.

**Theorem:** \( A_{n,k+1} = (k + 1)A_{n-1,k+1} + (n - k)A_{n-1,k} \)

**Proof.** Ask: How many \( n \)-permutations have \( k \) descents?

**LHS:** \( A_{n,k+1} \), of course!

**RHS:** Insert the number \( n \) into an \((n - 1)\)-permutation.

When \( n \) is inserted into an \((n - 1)\)-permutation with \( d \) descents, the resulting \( n \)-permutation either has

- \( d \) descents (If \( n \) inserted in a position that is a descent or at end.)
- \( d + 1 \) descents (If \( n \) inserted in a position that is not a descent.)

**Conclusion:** An \( n \)-perm with \( k \) descents can arise by inserting \( n \):

- into a perm with \( k \) existing descents in \((k + 1)A_{n-1,k+1}\) ways.
- into a perm with \( k - 1 \) existing descents in \((n - k)A_{n-1,k}\) ways.
Eulerian Numbers

The initial conditions $A_{n,1} = 1$ and $A_{n,n} = 1$ for all $n$ along with the recurrence

$$A_{n,k+1} = (k+1)A_{n-1,k+1} + (n-k)A_{n-1,k}$$

allow us to fill the chart:

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<tr>
<th>$n$</th>
<th>$A_{n,1}$</th>
<th>$A_{n,2}$</th>
<th>$A_{n,3}$</th>
<th>$A_{n,4}$</th>
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**Fact:** The Eulerian numbers satisfy the following identities.

$$A_{n,k} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n.$$

$$S(n, r) = \frac{1}{r!} \sum_{k=0}^{r} A_{n,k} \binom{n-k}{r-k}$$


**Inversion statistic**

*Definition:* Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation. An *inversion* is a pair $i < j$ such that $\pi_i > \pi_j$.

Define $\text{inv}(\pi)$ as the number of inversions in $\pi$.

**Example.** When $\pi = 416253$, $\text{inv}(\pi) = 7$ since $4 > 1$, $4 > 2$, $4 > 3$, $6 > 2$, $6 > 5$, $6 > 3$, $5 > 3$.

In a string diagram $\text{inv}(\pi) =$ number of crossings.

In a matrix diagram $\text{inv}(\pi)$, draw *Rothe diagram*:

$$
\begin{array}{cccccc}
\text{inv}(12) &= 0 & \text{inv}(123) &= \_ & \text{inv}(213) &= \_ & \text{inv}(312) &= \_ \\
\text{inv}(21) &= 1 & \text{inv}(132) &= \_ & \text{inv}(231) &= \_ & \text{inv}(321) &= \_
\end{array}
$$

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What are the possible values for $\text{inv}(\pi)$?

The inversion number is a good way to count how “far away” a permutation is from the identity.
Gaussian polynomials

**Definition:** \( b_{n,k} \) = number of \( n \)-permutations with \( k \) inversions.

**Theorem:** Let \( k \leq n \). Then \( b_{n+1,k} = b_{n+1,k-1} + b_{n,k} \)

**Proof.** Ask: How many \((n+1)\)-permutations have \( k \) descents?

**LHS:** \( b_{n+1,k} \), evidently!

**RHS:** Condition on the position of \((n+1)\).

The \((n+1)\)-perms with \( k \) descents and \((n+1)\) in the last position are in bijection with \______________________, and are counted by ___.

If \((n+1)\) is not in the last position, switch it with its right neighbor.

We recover an \((n+1)\)-permutation with \( k - 1 \) descents with the added condition that \___________________________.

Since \( k \leq n \), then every \((n+1)\)-permutation with \( k - 1 \) inversions satisfy this condition, (WHY?)

We conclude that there are \( b_{n+1,k-1} \) ways in which this can happen.
**Major index**

*Definition:* Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation.

Define $\text{maj}(\pi)$, the **major index** of $\pi$, to be sum of the descents of $\pi$.

[Named after Major Percy MacMahon. (British army, early 1900’s)]

**Example.** When $\pi = 416253$, $\text{maj}(\pi) = 9$ since the descents of $\pi$ are in positions 1, 3, and 5.

\[
\begin{align*}
\text{maj}(12) &= 0 & \text{maj}(123) &= \_ & \text{maj}(213) &= \_ & \text{maj}(312) &= \_ \\
\text{maj}(21) &= 1 & \text{maj}(132) &= \_ & \text{maj}(231) &= \_ & \text{maj}(321) &= \_
\end{align*}
\]

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What are the possible values for $\text{maj}(\pi)$?

The distribution of $\text{maj}(\pi)$ IS THE SAME AS the distribution of $\text{inv}(\pi)$!

A statistic that has the same distribution as $\text{inv}$ is called **Mahonian**.
There’s always more to learn!!!

*Theorem:* $\text{inv}$ and $\text{maj}$ are equidistributed on $S_n$.

Proofs exist using generating functions and using bijections.

- Find a bijection $f : S_n \rightarrow S_n$ such that $\text{maj}(\pi) = \text{inv}(f(\pi))$.

**References :**