Indistinguishable objects in indistinguishable boxes

When placing $k$ indistinguishable objects into $n$ indistinguishable boxes, what matters? 

We are partitioning the **integer** $k$ instead of the set $[k]$.

**Example.** What are the partitions of 6?

**Definition:** $P(k, i)$ is the number of partitions of $k$ into $i$ parts.

**Example.** We saw $P(6, 1) = 1$, $P(6, 2) = 3$, $P(6, 3) = 3$, $P(6, 4) = 2$, $P(6, 5) = 1$, and $P(6, 6) = 1$.

**Definition:** $P(k)$ is the number of partitions of $k$ into any number of parts.

**Example.** $P(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11$. 
**Question:** In how many ways can we place \( k \) objects in \( n \) boxes?

<table>
<thead>
<tr>
<th>Distributions of ( k ) objects in ( n ) boxes</th>
<th>Restrictions on # objects received</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>( \leq 1 )</td>
</tr>
<tr>
<td>distinct distinct</td>
<td>( n^k )</td>
</tr>
<tr>
<td>identical distinct</td>
<td>( \binom{n}{k} )</td>
</tr>
<tr>
<td>distinct identical</td>
<td>( \sum S(k, i) )</td>
</tr>
<tr>
<td>identical identical</td>
<td>( \sum P(k, i) )</td>
</tr>
</tbody>
</table>

\( P(k, n) \) counts ways to place \( k \) identical obj. into \( n \) identical boxes.

How many ways to distribute identical objects into identical boxes

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions?
Example. Suppose that in this class, 14 students play soccer and 17 students play basketball. How many students play a sport?

Solution.

Let $S$ be the set of students who play soccer and $B$ be the set of students who play basketball. Then, $|S \cup B| = |S| + |B|$.
Principle of Inclusion-Exclusion

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) (\( \mathcal{U} \) for universe) and the sets \( A_i \) are **pairwise disjoint**, we have \(|A| = |A_1| + \cdots + |A_k|\).

When \( A = A_1 \cup \cdots \cup A_k \subset \mathcal{U} \) and the \( A_i \) are **not** pairwise disjoint, we must apply the principle of inclusion-exclusion to determine \(|A|\):

\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|
\]

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\]

\[
|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots
\]

It may be more convenient to apply inclusion/exclusion where the \( A_i \) are **forbidden** subsets of \( \mathcal{U} \), in which case ________________.
Example. Find the number of integers between 1 and 1000 that are not divisible by 5, 6, or 8.

Solution. Let \( \mathcal{U} = \{ n \in \mathbb{Z} \text{ such that } 1 \leq n \leq 1000 \} \).
Let \( A_1 \subset \mathcal{U} \) be the multiples of 5, \( A_2 \subset \mathcal{U} \) be the multiples of 6, and \( A_3 \subset \mathcal{U} \) be the multiples of 8. We want \( |\mathcal{U}| - |A_1 \cup A_2 \cup A_3| \).

In words, \( A_1 \cap A_2 \) is the set of integers which are \( A_1 \cap A_3 \) is \( A_2 \cap A_3 \) is and \( A_1 \cap A_2 \cap A_3 \) is the set of integers which are

Now calculate: \( |A_1| = \quad |A_2| = \quad |A_3| = \)
\( |A_1 \cap A_2| = \quad |A_1 \cap A_3| = \quad |A_2 \cap A_3| = \)
\( |A_1 \cap A_2 \cap A_3| = \)

And finally: So \( |\mathcal{U}| - |A_1 \cup A_2 \cup A_3| = \)
Combinations with Repetitions

Quick review

1. How many ways are there to choose \( k \) elements out of the set \( \{1 \cdot a_1, 1 \cdot a_2, \ldots, 1 \cdot a_n\} \)?

2. How many ways are there to choose \( k \) elements out of the set \( \{k \cdot a_1, k \cdot a_2, \ldots, k \cdot a_n\} \)? (really \( \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\} \))

What we would like to calculate is:

In how many ways can we choose \( k \) elements out of an arbitrary multiset?

Now, it’s as easy as PIE.
Example. Determine the number of 10-combinations of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

**Game plan:** Let $\mathcal{U}$ be the set of 10-combs of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Use PIE to remove the 10-combs that violate the conditions of $S$.

Define $A_1$ to be 10-combs that include at least ___ $a$’s.
Define $A_2$ to be 10-combs that include at least ___ $b$’s.
Define $A_3$ to be 10-combs that include at least ___ $c$’s.

**In words,** $A_1 \cap A_2$ are those 10-combs that
$A_1 \cap A_3$: 
$A_2 \cap A_3$: 
$A_1 \cap A_2 \cap A_3$

**Now calculate:** $|\mathcal{U}| = \quad |A_1| = \quad |A_2| = \quad |A_3| = \quad |A_1 \cap A_2| = \quad |A_1 \cap A_3| = \quad |A_2 \cap A_3| = \quad |A_1 \cap A_2 \cap A_3| = \quad$

**And finally:** So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| = \quad$
At a party, 10 gentlemen check their hats. They “have a good time”, and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his own hat?

This is a derangement of ten objects.

**Definition:** An *n*-derangement is an *n*-permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1$, $p_2 \neq 2$, $\cdots$, $p_n \neq n$.

**Note:** A derangement is a permutation without fixed points $\pi(i) = i$.

**Notation:** We let $D_n$ be the number of all *n*-derangements.

When you see $D_n$, think combinatorially: “The number of ways to return *n* hats to *n* people so no one gets his/her own hat back”
Calculating the number of derangements

Example. Calculate $D_n$.

Solution. Let $\mathcal{U}$ be the set of all $n$-permutations. Remove bad permutations using PIE. For all $i$ from 1 to $n$, define $A_i$ to be $n$-perms where $p_i = i$.

In words, $A_i \cap A_j$ are $n$-perms where $A_i \cap A_j \cap A_k$ are $n$-perms where In general, $A_{i_1} \cap \cdots \cap A_{i_k}$ are $n$-perms with $p_{i_1} = i_1$, $\cdots$, $p_{i_k} = i_k$.

Now calculate: $|\mathcal{U}| = |A_1| = |A_2| =$

For all $i$ and $j$, $|A_i \cap A_j| =$

When intersecting $k$ sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots$

Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \cdots \cup A_n| =$
Randomly returning hats

Upon simplification, we see
\[
D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}0!
\]
\[
= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}
\]
\[
= n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]
\]

Recall: Taylor series expansion of \( e^x \):
\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.
\]
Plug in \( x = -1 \) and truncate after \( n \) terms to see that
\[
e^{-1} \approx \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]
\]

Conclusion: If \( n \) people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is \( D_n/n! \), which is approximately \( 1/e \approx 37\% \).
Recall: The combinatorial interpretation of $D_n$ is: “The number of ways to return $n$ hats to $n$ people so no one gets his/her own hat back”

Example. Prove the following recurrence relation for $D_n$ combinatorially.

$$D_n = (n - 1)(D_{n-2} + D_{n-1})$$
A formula for Stirling numbers

Recall: $S(n, k) = \binom{n}{k}$ is the number of partitions of the set $[n]$ into exactly $k$ parts, and $k!S(n, k)$ is the number of onto functions $[n] \rightarrow [k]$.

**Question:** What is a formula for $S(n, k)$?

**Solution.** We will find the number of surjections from $[n]$ to $[k]$. Use PIE with $\mathcal{U} =$ set of all functions from $[n]$ to $[k]$. We will remove the “bad” functions where the range is not $[k]$.

Define $A_i$ be the set of functions $f : [n] \rightarrow [k]$ where $i$ is not “hit”.

**In words**, $A_{i_1} \cap \cdots \cap A_{i_j}$ are functions where none of $i_1$ through $i_j$ are elements of the image.

**We calculate:** $|\mathcal{U}| = k^n$, $|A_i| = (k - 1)^n$, $|A_i \cap A_j| = (k - 2)^n$.

When intersecting $j$ sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$.

Therefore, $k!S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n$; we conclude $S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n$. 