What is a Combinatorial Proof?

**Definition:** A **combinatorial interpretation** of a numerical quantity is a set of combinatorial objects that is counted by the quantity.

**Example.** We can choose \( k \) objects out of \( n \) total objects in \( \binom{n}{k} \) ways. Use this fact “backwards” by interpreting an occurrence of \( \binom{n}{k} \) as the number of ways to choose \( k \) objects out of \( n \).

This leads to my favorite kind of proof:

**Definition:** A **combinatorial proof** of an identity \( X = Y \) is a proof by counting (!). You find a set of objects that can be interpreted as a combinatorial interpretation of both the **left hand side (LHS)** and the **right hand side (RHS)** of the equation. As both sides of the equation count the same set of objects, they must be equal!

- It is important to get the set of objects right.
- To do this, you must ask a good question: “In how many ways...”
Example. Prove Equation (2.2): For $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$.
(We already know a bijective proof of this fact.)

**Analytic Proof:**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

**Combinatorial Proof:**

**Question:** In how many ways can we adopt $k$ of $n$ cats available for adoption at the animal shelter?

**Answer 1:** Choose $k$ of the $n$ cats to adopt in $\binom{n}{k}$ ways.

**Answer 2:** Choose $n - k$ of the $n$ cats to NOT adopt in $\binom{n}{n-k}$ ways.

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □
Another Simple Combinatorial Proof

Example. Prove Equation (2.4): \( k\binom{n}{k} = n\binom{n-1}{k-1} \).

**Analytic Proof:**

**Combinatorial Proof:**

*Question:* In how many ways can we choose from \( n \) club members a committee of \( k \) members with a chairperson?

*Answer 1:*

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. \( \square \)
Example. Prove Theorem 2.2.1: \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \).

**Combinatorial Proof:**

**Question:** In how many ways can we choose \( k \) flavors of ice cream if \( n \) different choices are available?

**Answer 1:**

**Answer 2:**

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □
Summing Binomial Coefficients

Example. Prove Equation (2.3): \( \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \).

**Analytic Proof:**

**Combinatorial Proof:**

**Question:** How many subsets of \( \{1, 2, \ldots, n\} \) are there?

**Answer 1:** Condition on how many elements are in a subset.

**Answer 2:**

Because the two quantities count the same set of objects in two different ways, the two answers are equal.  

—Worksheet—