Theorem. Let $G$ be a planar graph. There exists a proper 6-coloring of $G$.

Proof. Let $G$ be a the smallest planar graph (by number of vertices) that has no proper 6-coloring.

By Theorem 8.1.7, there exists a vertex $v$ in $G$ that has degree five or less. $G \setminus v$ is a planar graph smaller than $G$, so it has a proper 6-coloring.

Color the vertices of $G \setminus v$ with six colors; the neighbors of $v$ in $G$ are colored by at most five different colors.

We can color $v$ with a color not used to color the neighbors of $v$, and we have a proper 6-coloring of $G$, contradicting the definition of $G$. 
The Five Color Theorem

**Theorem.** Let $G$ be a planar graph. There exists a proper 5-coloring of $G$.

**Proof.** Let $G$ be the smallest planar graph (by number of vertices) that has no proper 5-coloring.

By Theorem 8.1.7, there exists a vertex $v$ in $G$ that has degree five or less. $G \setminus v$ is a planar graph smaller than $G$, so it has a proper 5-coloring.

Color the vertices of $G \setminus v$ with five colors; the neighbors of $v$ in $G$ are colored by at most five different colors.

If they are colored with only four colors, we can color $v$ with a color not used to color the neighbors of $v$, and we have a proper 5-coloring of $G$, contradicting the definition of $G$.
The Kempe Chains Argument

Otherwise the neighbors of \( v \) are all colored differently. We will work to modify the coloring on \( G \setminus v \) so that only four colors are used.

Consider the subgraphs \( H_{1,3} \) and \( H_{2,4} \) of \( G \setminus v \) constructed as follows: Let \( V_{1,3} \) be the set of vertices in \( G \setminus v \) colored with colors 1 or 3. Let \( V_{2,4} \) be the set of vertices in \( G \setminus v \) colored with colors 2 or 4. Let \( H_{1,3} \) be the induced subgraph of \( G \) on \( V_{1,3} \). (Define \( H_{2,4} \) similarly)
The Kempe Chains Argument

**Definition:** A Kempe chain is a path in $G \setminus v$ between two non-consecutive neighbors of $v$ such that the colors on the vertices of the path alternate between the colors on those two neighbors.

In the example above, $3 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 1$ is a Kempe chain: the colors alternate between red and green and $1, 3$ not consecutive.

Either $v_1$ and $v_3$ are in the same component of $H_{1,3}$ or not. If they are, there is a Kempe chain between $v_1$ and $v_3$. If they are not, (say $v_1$ is in component $C_1$ and $v_3$ is in $C_3$) then swap colors 1 and 3 in $C_1$. (Here we show $C_2$ and $C_4$)
The Kempe Chains Argument

**Claim.** This remains a proper coloring of $G \setminus v$.

**Proof.** We need to check that the recoloring does not have two like-colored vertices adjacent.

In $C_1$, there are only vertices of color 1 and 3 and recoloring does not change that no two adjacent vertices are colored differently.

And, by construction, no vertex adjacent to a vertex in $C_1$ is colored 1 or 3. This is true before AND after recoloring.  

\[\square\]
The Kempe Chains Argument

So **either** there is a Kempe chain between \( v_1 \) and \( v_3 \) **or** we can swap colors so that \( v \)’s neighbors are colored only using four colors.

Similarly, **either** there is a Kempe chain between \( v_2 \) and \( v_4 \) **or** we can swap colors to color \( v \)’s neighbors with only four colors.

**Question.** Can we have both a \( v_1-v_3 \) and a \( v_2-v_4 \) Kempe chain?

There are no edge crossings in the graph drawing, so there would exist a vertex_______________________________.

This cannot exist, so it must be possible to swap colors and be able to place a fifth color on \( v \), contradicting the definition of \( G \).
Modifications of Graphs

**Definition:** **Deletion**

\[ G \setminus v \ (G \text{ delete } v) \]: Remove \( v \) from the graph and all incident edges.

\[ G \setminus e \ (G \text{ delete } e) \]: Remove \( e \) from the graph.

**Definition:** **Contraction**

\[ G/e \ (G \text{ contract } e) \]: If \( e = vw \), coalesce \( v \) and \( w \) into a super-vertex adjacent to all neighbors of \( v \) and \( w \). \([This \ may \ produce \ a \ multigraph.]\]

**Definition:** A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from \( G \) by a sequence of edge deletions and/or edge contractions. \([“Minor” \ suggests \ smaller: \ H \ is \ smaller \ than \ G.]\)

**Note.** Any subgraph of \( G \) is also a minor of \( G \).
Modifications of Graphs

**Definition:** A **subdivision** of an edge $e$ is the replacement of $e$ by a path of length at least two. [Like adding vertices in the middle of $e$.]

**Definition:** A **subdivision** of a graph $H$ is the result of zero or more sequential subdivisions of edges of $H$.

**Note.** If $G$ is a subdivision of $H$, then $G$ is at least as large as $H$.

**Note.** If $G$ is a subdivision of $H$, then $H$ is a minor of $G$. (Contract any edges that had been subdivided!)

**Note.** The converse is not necessarily true.
Kuratowski’s Theorem

Theorem. Let $H$ be a subgraph of $G$. If $H$ is nonplanar, then $G$ is nonplanar.

Theorem. Let $G$ be a subdivision of $H$. If $H$ is nonplanar, then $G$ is nonplanar.

Corollary. If $G$ contains a subdivision of a nonplanar graph, then $G$ is nonplanar.

Theorem. (Kuratowski, 1930) A graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

Theorem. (Kuratowski variant) A graph $G$ is planar if and only if neither $K_5$ nor $K_{3,3}$ is a minor of $G$. 
Kuratowski’s Theorem

To prove that a graph $G$ is planar, find a planar embedding of $G$. To prove that a graph $G$ is non-planar, (a) find a subgraph of $G$ that is isomorphic to a subdivision of $K_5$ or $K_{3,3}$, or (b) successively delete and contract edges of $G$ to show that $K_5$ or $K_{3,3}$ is a minor of $G$.

Practice on the Petersen graph. (Here, have some copies!)