Planarity

Up until now, graphs have been completely abstract. In Topological Graph Theory, it matters how the graphs are drawn.

- Do the edges cross?
- Are there knots in the graph structure?

**Definition:** A **drawing** of a graph $G$ is a pictorial representation of $G$ in the plane as points and line segments. The line segments must be **simple curves**, which means no intersections are allowed.

**Definition:** A **plane drawing** of a graph $G$ is a drawing of the graph in the plane with no crossings. Otherwise, $G$ is **nonplanar**.

**Definition:** A **planar graph** is a graph that has a plane drawing.

**Example.** $K_4$ is a planar graph because it is a plane drawing of $K_4$. 
Vertices, Edges, and Faces

Definition: In a plane drawing, edges divide the plane into regions, or faces.

There will always be one face with infinite area. This is called the outside face.

Notation. Let \( p = \# \) of vertices, \( q = \# \) of edges, \( r = \# \) of regions. Compute the following data:

<table>
<thead>
<tr>
<th>Graph</th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cube</td>
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<tr>
<td>Octahedron</td>
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<tr>
<td>Dodecahedron</td>
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<tr>
<td>Icosahedron</td>
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</tbody>
</table>

In 1750, Euler noticed that __________ in each of these examples.
**Euler’s Formula**

**Theorem 8.1.1** (Euler’s Formula) If $G$ is connected, then in a plane drawing of $G$, $p - q + r = 2$.

**Proof** (by induction on the number of cycles)

**Base Case:** If $G$ is a tree, there is one region, so

$$p - q + r = p - (p - 1) + 1 = 2.$$ 

**Inductive Step:** Suppose that for all plane drawings with fewer than $k$ cycles, $p - q + r = 2$, we wish to prove that in a plane drawing of a graph $G$ with exactly $k$ cycles, $p - q + r = 2$ also holds.

Let $C$ be a cycle in $G$. Let $e$ be any edge in $C$, then $e$ is adjacent to two different regions, one inside $C$ and one outside $C$.

$G \setminus e$ has fewer cycles than $G$, and one fewer region. The inductive hypothesis holds for $G \setminus e$, giving
Maximal Planar Graphs

A graph with “too many” edges isn’t planar; how many is too many?

**Goal:** Find a numerical characterization of “too many”

**Definition:** A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

**Examples.** Octahedron \( K_4 \) \( K_5 \setminus e \)

What do we notice about these graphs?
Numerical Conditions on Planar Graphs

- Every face of a maximal planar graph is a triangle!

If not,

*Theorem 8.1.2.* If $G$ is *maximal planar* and $p \geq 3$, then $q = 3p - 6$.

*Proof.* Consider any plane drawing of $G$.
Let $p = \# \text{ of vertices}$, $q = \# \text{ of edges}$, and $r = \# \text{ of regions}$.
We will count the number of face-edge incidences in two ways:
From a face-centric POV, the number of face-edge incidences is
From an edge-centric POV, the number of face-edge incidences is
Now substitute into Euler’s formula:

Do we need $p \geq 3$?
Corollary 8.1.3. Every planar graph with $p \geq 3$ vertices has at most $3p - 6$ edges.

- Start with any planar graph $G$ with $p$ vertices and $q$ edges.
- Add edges to $G$ until it is maximal planar. (with $Q \geq q$ edges.)
- This resulting graph satisfies $Q = 3p - 6$; hence $q \leq 3p - 6$.

Theorem 8.1.4. The graph $K_5$ is not planar.

Proof.
Numerical Conditions on Planar Graphs

Definition: The girth $g(G)$ of a graph $G$ is the smallest cycle size.

Example.

Theorem 8.1.5.* If $G$ is planar with girth $\geq 4$, then $q \leq 2p - 4$.

Proof. Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$ 

Therefore, $q \leq 2p - 4$.

Theorem 8.1.5. If $G$ is planar and bipartite, then $q \leq 2p - 4$.

Theorem 8.1.6. $K_{3,3}$ is not planar.

Theorem 8.1.7. Every planar graph has a vertex with degree $\leq 5$.

Proof.
Definition: Given a plane drawing of a planar graph $G$, the **dual graph** $D(G)$ of $G$ is a graph with vertices corresponding to the regions of $G$. Two vertices are connected by an edge each time the two regions share an edge as a border.

- The dual graph of a simple graph may not be simple.
- Two regions may be adjacent multiple times.
- $G$ and $D(G)$ have the same number of edges.

Definition: A graph $G$ is **self-dual** if $G$ is isomorphic to $D(G)$. 
Maps

**Definition:** A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a **normal map**.

**Definition:** In a map, the regions are called **countries**. Countries may share several edges.

**Definition:** A **proper coloring** of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

**Question.** How many colors are necessary to properly color a map?
Lemma 8.2.2. If $M$ is a map that is a union of simple closed curves, the regions can be colored by two colors.

Proof. Color the regions $R$ of $M$ as follows:

\[
\begin{cases}
    \text{black} & \text{if } R \text{ is enclosed in an odd number of curves} \\
    \text{white} & \text{if } R \text{ is enclosed in an even number of curves}
\end{cases}
\]

This is a proper coloring of $M$. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.
The Four Color Theorem

**Lemma 8.2.6.** (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

**Proof.** Very hard.

★ This is the wrong object ★

**Theorem.** If $G$ is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

- Every face of $G$ is a triangle $\rightsquigarrow$
- $G$ is connected $\rightsquigarrow$
- $G$ is planar $\rightsquigarrow$
The Four Color Theorem

Assuming Lemma 8.2.6,

- $G$ is maximal planar $\Rightarrow$ $D(G)$ is a normal map
- $\Rightarrow$ countries of $D(G)$ 4-colorable
- $\Rightarrow$ vertices of $G$ 4-colorable
- $\Rightarrow$ $\chi(G) \leq 4$

This proves

**Theorem 8.2.8** If $G$ is maximal planar, then $\chi(G) \leq 4$.

Since every planar graph is a subgraph of a maximal planar graph, Lemma C implies:

**Theorem 8.2.9.** If $G$ is a planar graph, then $\chi(G) \leq 4$.

★ History ★