Path graph \( P_n \): The path graph \( P_n \) has \( n + 1 \) vertices, 
\[ V = \{v_0, v_1, \ldots, v_n\} \] and \( n \) edges, 
\[ E = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}. \]

The length of a path is the number of edges in the path.

Cycle graph \( C_n \): The cycle graph \( C_n \) has \( n \) vertices, 
\[ V = \{v_1, \ldots, v_n\} \] and \( n \) edges, 
\[ E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}. \]

We often try to find and/or count paths and cycles in a graph.
Complete graph $K_n$: The complete graph $K_n$ has $n$ edges, $V = \{v_1, \ldots, v_n\}$ and has an edge connecting every pair of distinct vertices, for a total of $\binom{n}{2}$ edges.

Definition: a bipartite graph is a graph where the vertex set can be broken into two parts such that there are no edges between vertices in the same part.

Complete bipartite graph $K_{m,n}$: The complete bipartite graph $K_{m,n}$ has $m + n$ vertices $V = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ and an edge connecting each $v$ vertex to each $w$ vertex.
Families of Graphs

- **Wheel graph** $W_n$: The wheel graph $W_n$ has $n+1$ vertices $V = \{v_0, v_1, \ldots, v_n\}$. Arrange and connect the last $n$ vertices in a cycle (the rim of the wheel). Place $v_0$ in the center (the hub), and connect it to every other vertex.

- **Star graph** $St_n$: The star graph $St_n$ has $n+1$ vertices $V = \{v_0, v_1, \ldots, v_n\}$ and $n$ edges $\{v_0v_1, v_0v_2, \ldots, v_0v_n\}$.

- **Cube graph** $\Box_n$: The cube graph in $n$ dimensions, $\Box_n$, has $2^n$ vertices. We index the vertices by binary numbers of length $n$. We connect two vertices when their binary numbers differ by exactly one digit.
Two graphs we will see on a consistent basis are:

Petersen graph $P$

Grötzsch graph $Gr$
Definition: The **platonic solids** are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. They are the only regular convex polyhedra made of regular polygons.

Definition: The **Schlegel diagram** of a polyhedron is a planar 2D graph that represents a 3D object, where vertices of the graph represent vertices of the polyhedron, and edges of the graph represent the edges of the polyhedron.

The **Platonic graphs** are the Schlegel diagrams of the five platonic solids.
When are two graphs the same?

Two graphs $G_1$ and $G_2$ are **equal** ($G_1 = G_2$) if they have the exact same vertex sets and edge sets. The graphs $G_1$ and $G_2$ are **isomorphic** ($G_1 \cong G_2$) if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ such that $v_i v_j$ is an edge of $G_1$ iff $\varphi(v_i) \varphi(v_j)$ is an edge of $G_2$.

In this course, we will spend a large amount of time trying to figure out whether two given graphs are the same.

Side note: For a graph, the set of homomorphisms (isomorphisms into itself) is a measure of the symmetry of the graph. Ex. ⭐️
The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ can mean two different things:

- When the vertex sets are different, the **(disjoint) union** $H$ of $G_1$ and $G_2$ is formed by placing the graphs side by side. In this case, $H = (V_1 \cup V_2, E_1 \cup E_2)$.

- When the vertex sets are the same, then the **(edge) union** $H$ of $G_1$ and $G_2$ contains every edge of both $E_1$ and $E_2$. In this case, $H = (V, E_1 \cup E_2)$.

The **complement** $G^c$ or $\overline{G}$ of a graph $G = (V, E)$ is a graph with the same vertex set. Its edge set contains all edges **NOT** in $G$.

If $G = (V, E_1)$ and $G^c = (V, E_2)$, then $E_1 \cup E_2 = E(K_n)$, and $E_1 \cap E_2 = \emptyset$. 
Subgraphs

A subgraph $H$ of a graph $G$ is a graph where every vertex of $H$ is a vertex of $G$, and that every edge of $H$ is an edge of $G$.

★ If edge $e$ of $G$ is in $H$, then the endpoints of $e$ must also be in $H$.

A subgraph $H$ is a proper subgraph if $H \neq G$.

If $G_1$ and $G_2$ are two graphs, we say that $G_1$ contains $G_2$ if there exists a subgraph $H$ of $G_1$ such that $H$ is isomorphic to $G_2$.

Example. Show that the wheel $W_6$ contains a cycle of length 3, 4, 5, 6, and 7.
Induced Subgraphs

An **induced subgraph** $H$ of a graph $G$ is determined by a set of vertices $W \subseteq V(G)$. Define $H$ to have as its vertex set, $W$, and as its edge set, the set of edges from $E(G)$ between vertices in $W$.

Induced subgraphs of $G$ are always subgraphs of $G$, but not vice versa.