The Enumeration of Fully Commutative Affine Permutations

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Permutations and affine permutations

A Coxeter group is a group with
- Generators: \{s_1, s_2, \ldots, s_n\}
- Relations: \( r_i^2 = 1 \), \( (s_ils_j)^{m_{ij}} = 1 \) where \( m_{ij} \) is 2 or \( \infty \)
- \( m_{ij} = 2 \): \( (s_is_j)^2 = 1 \) \( \iff \ s_i s_j = s_j s_i \) (they commute)
- \( m_{ij} = 3 \): \( (s_is_j)^3 = 1 \) \( \iff \ s_i s_j s_i = s_j s_i s_j \) (braid relation)

An element’s length is the number of generators in its shortest expression.

(Finite) Permutations \( S_n \)

- Write an element \( w \in S_n \) in 1-line notation as a permutation of \( \{1, \ldots, n\} \).
- Generators transpose adjacent entries: \( s_i : (i) \rightarrow (i+1) \).
- Example. In \( S_4 \), \( s_1 s_2 = 214536 \).

Affine Permutations \( \tilde{S}_n \)

- Write an element \( w \in \tilde{S}_n \) in 1-line notation as a permutation of \( \mathbb{Z} \).
- Generators infinitely many pairs of entries:
  - \( \{(i), (i+1)\} \), \( \{(i), (i-1)\} \), \( \ldots \)
  - \( w \) is defined by the window \( [w(1), w(2), \ldots, w(n)] \).
- Example. In \( \tilde{S}_3 \), \( \tilde{s_1} \tilde{s_2} \tilde{s_1} = 20134567910 \).

Fully commutative elements

An element in a Coxeter group is fully commutative if it has only one reduced expression (up to commutation relations).

- No BRAIDS ALLOWED!

Example. In \( S_2 \), \( s_1 s_2 s_1 \) is not fully commutative because \( s_1 s_2 s_1 = s_1 s_2 \) \( \neq s_2 s_1 \).

- GOAL: Enumerate fully commutative affine permutations by Coxeter length.

Game plan

- (Green, 2002) characterizes when \( w \in S_n \) is fully commutative.
  - \( w \) is fully commutative \( \iff \ w \) is 321-avoiding.
  - For example, \( w = 4132 \) is 321-avoiding.
  - \( \emptyset \) is 321-avoiding.

- Write \( w = w^+ w^- \), where \( w^+ \in S_n \) and \( w^- \in \tilde{S}_n \).
  - For \( w = [-1; 2, 3, 4, 5, 0] \in \tilde{S}_5 \), \( w^+ = [-1, -2, 0, 3, 4, 5] \) and \( w = [-1, 3, 6, 4, 5, 2] \).
  - Determine which \( w \) is 321-avoiding.
  - (Use algorithm 1-avoiding).

- Determine which \( w \) satisfies \( w^+ w^- \) still 321-avoiding.
  - (Depends on the structure of \( w^+ w^- \); partition into long and short elements.)

Data

\[
\begin{align*}
\ell(q) & = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + \cdots \\
\ell(q) & = 1 + 4q + 10q^2 + 18q^3 + 16q^4 + \cdots \\
\ell(q) & = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + 50q^5 + 50q^6 + \cdots \\
\ell(q) & = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \cdots \\
150q^2 & + 156q^3 + 156q^4 + 156q^5 + 150q^6 + 158q^7 + \cdots \\
150q^2 & + 156q^3 + 156q^4 + 156q^5 + 150q^6 + 158q^7 + \cdots \\
\ell(q) & = 1 + 7q + 28q^2 + 77q^3 + 266q^4 + 364q^5 + 427q^6 + 462q^7 + 483q^8 + 490q^9 + 490q^10 + 480q^11 + 480q^12 + \cdots 
\end{align*}
\]

Notice:
- The coefficients eventually repeat.
- For prime \( n \), the period is 1.

Combinatorial Models for \( S_n / S_2 \)

- (James and Kerber, 1981) Interpret \( w^+ \in S_n \) as \( \ell(n) \) runiners.
  - Place integers in \( n \) runiners.
  - \( \text{Circled entries: beads} \)
  - \( \text{Empty entries: gaps} \)
  - Bijection: Create an abacus where each runner has a lowest bead at \( w^- \).
  - Example. \([-4, -3, 7, 10]\).

Abacus diagrams

We use a normalized abacus diagram: shifts all beads so that the first gap is in position \( n + 1 \); this map is invertible.

Theorem. (H-J ’09) Given a normalized abacus for \( w \in S_n / S_2 \), where the last bead occurs in position \( i \), \( w^+ \) is lowest beads in runners only occur in fully commutative \( \{1, \ldots, n\} \cup \{n+1, \ldots, n+i\} \).

Ideas:
- Lowest beads in runners correspond to entries in base window.
- with only low and high entries
  - with low and high entries
  - with medium entries as well

Short elements versus long elements

Partition \( S_n \) into long and short elements:

- Short elements
  - Lowest bead in position \( i \leq 2n \)
  - Hard to count
- Long elements
  - Lowest bead in position \( i > 2n \)
  - Come in infinite families
  - Easy to count
  - Explain the periodicity

References