The Enumeration of Fully Commutative Affine Permutations

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Permutations and affine permutations

A **Coxeter group** is a group with

- Generators: $\{s_1, s_2, \ldots, s_n\}$
- Relations: $s_i^2 = 1$, $\quad (s_i s_j)^{m_{i,j}} = 1$ where $m_{i,j} \geq 2$ or $= \infty$
  - $m_{i,j} = 2$: $(s_i s_j)(s_i s_j) = 1 \quad \longrightarrow \quad s_i s_j = s_j s_i$ (they commute)
  - $m_{i,j} = 3$: $(s_i s_j)(s_i s_j)(s_i s_j) = 1 \rightarrow s_i s_j s_i = s_j s_i s_j$ (braid relation)
- An element's **length** is the number of generators in its shortest expression.

(Finite) n-permutations S_n

- Write an element $w \in S_n$ in 1-line notation as a permutation of $\{1, \ldots, n\}$.
- Generators transpose adjacent entries: $s_i : (i) \leftrightarrow (i+1)$.
- *Example.* In S_6, $s_3 s_4 s_1 = 2\,1\,4\,5\,3\,6$.

Affine n-Permutations $\tilde{S}_n$

- Write an element $\tilde{w} \in \tilde{S}_n$ in 1-line notation as a permutation of $\mathbb{Z}$.
- Generators transpose **infinitely many** pairs of entries:
  $s_i : (i) \leftrightarrow (i+1) \ldots (n+i) \leftrightarrow (n+i+1) \ldots (-n+i) \leftrightarrow (-n+i+1) \ldots$
- $\tilde{w}$ is defined by the **window** $[\tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(n)]$.
- *Example.* In $\tilde{S}_4$, $s_1 s_0 = [0, 1, 3, 6]$:

	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9) \cdots$
s_1	$\cdots -4$	-2	-3	-1	0	2	1	3	4	6	5	7	8	$10 \cdots$
s_0	$\cdots -3$	-4	-2	-1	1	0	2	3	5	4	6	7	9	$8 \cdots$
$s_1 s_0$	$\cdots -2$	-4	-3	-1	2	0	1	3	6	4	5	7	10	$8 \cdots$

Fully commutative elements

An element in a Coxeter group is **fully commutative** if it has only one reduced expression (up to commutation relations).

NO BRAIDS ALLOWED!

Example. In S_4, $s_1 s_2 s_3 s_1$ is **not fully commutative** because

$$s_1 s_2 s_3 s_1 \overset{\text{OK}}{=} s_1 s_2 s_1 s_3 \overset{\text{BAD}}{=} s_2 s_1 s_2 s_3$$

GOAL: Enumerate fully commutative affine permutations by Coxeter length.

Game plan

★ (Green, 2002) characterizes when $\tilde{w} \in \tilde{S}_n$ is fully commutative.

$\tilde{w}$ is fully commutative $\iff$ $\tilde{w}$ is 321-avoiding.

For example, $\tilde{w} = [-4, -1, 1, 14]$ is **NOT** fully commutative:

	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9) \cdots$
$\tilde{w}$	$\cdots 6$	-8	-5	-3	10	-4	-1	1	14	0	3	5	18	$4 \cdots$

★ Write $\tilde{w} = w^0 w$, where $w^0 \in \tilde{S}_n / S_n$ and $w \in S_n$.
For $\tilde{w} = [-11, 20, -3, 4, 11, 0] \in \tilde{S}_6$, $w^0 = [-11, -3, 0, 4, 11, 20]$ and $w = [1, 3, 6, 4, 5, 2]$.
(w^0 determines the entries, w determines the order.)

★ Determine which w^0 are 321-avoiding.
(*Use abacus diagrams.*)

★ Determine which w satisfy $w^0 w$ still 321-avoiding.
(Depends on the structure of w^0; partition into long and short elements.)

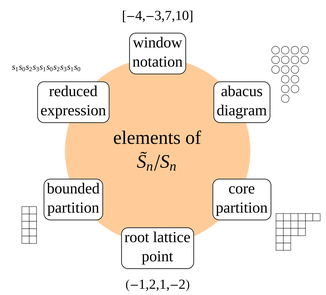
Data

$$f_3(q) = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + 6q^5 + \cdots$$
$$f_4(q) = 1 + 4q + 10q^2 + 16q^3 + 18q^4 + 16q^5 + 18q^6 + \cdots$$
$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + 50q^5 + 50q^6 + 50q^7 + 50q^8 + \cdots$$
$$f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + 150q^7 + 156q^8 + 152q^9 + 156q^{10} + 150q^{11} + 158q^{12} + 150q^{13} + 156q^{14} + 152q^{15} + 156q^{16} + 150q^{17} + 158q^{18} + \cdots$$
$$f_7(q) = 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + 490q^{10} + 490q^{11} + 490q^{12} + 490q^{13} + \cdots$$

Notice:

- The coefficients eventually repeat.
- For prime n, the period is 1.

Combinatorial Models for $\tilde{S}_n/S_n$

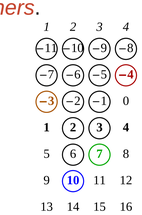


Abacus diagrams

(James and Kerber, 1981) Interpret $w^0 = [w_1, \ldots, w_n] \in \tilde{S}_n/S_n$ as follows:

- Place integers in n *runners*.
- Circled entries: *beads*
- Empty entries: *gaps*
- Bijection: Create an abacus where each runner has a lowest bead at w_i

Example: $[-4, -3, 7, 10]$

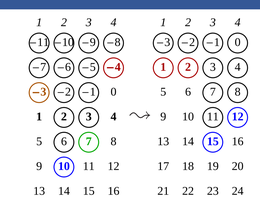


Normalized abacus diagrams

We use a *normalized* abacus diagram; shifts all beads so that the first gap is in position $n+1$; this map is invertible.

Theorem. (H–J '09) Given a normalized abacus for $w^0 \in \tilde{S}_n/S_n$, where the last bead occurs in position i,

w^0 is fully commutative $\iff$ lowest beads in runners only occur in $\{1, \ldots, n\} \cup \{i-n+1, \ldots, i\}$



Idea: Lowest beads in runners correspond to entries in base window.

	$w(-n+1)$ $w(-n+2)$ … $w(-1)$ $w(0)$	$w(1)$ $w(2)$ … $w(n-1)$ $w(n)$	$w(n+1)$ $w(n+2)$ … $w(2n-1)$ $w(2n)$
with only low and high entries	lo lo … hi hi	lo lo … hi hi	lo lo … hi hi
with medium entries as well	lo lo med hi **hi**	lo lo **med** hi hi	**lo** lo med hi hi

Short elements versus long elements

Partition $\tilde{S}_n$ into long and short elements:

Short elements

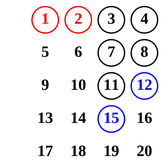
- Lowest bead in position $i \leq 2n$
- Hard to count

Long elements

- Lowest bead in position $i > 2n$
- Come in infinite families
- **Easy to count**
- *Explain the periodicity*

Enumerating long elements

For long elements $\tilde{w} \in \tilde{S}_n$, the base window for w^0 is $[a, a, \ldots, a, b, b, \ldots, b]$ where $1 \leq a \leq n$, and $n+2 \leq b$.



Question: Which $w \in S_n$ can be multiplied into w^0?

- We can not invert any pairs of a's, nor any pairs of b's. (Would create a 321-pattern with an adjacent window)
- Only possible to *intersperse* the a's and the b's.

How many ways to intersperse (k) a's and $(n-k)$ b's? $\binom{n}{k}$

BUT: We must also keep track of the *length* of these permutations.
This is counted by the q-binomial coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}$

Long elements explain periodicity

- Enumerate by length all possible w^0 with (k) a's and $(n-k)$ b's.
- Combine the Coxeter lengths by $\ell(\tilde{w}) = \ell(w^0) + \ell(w)$.

Theorem. (H–J '09) For a fixed $n \geq 0$, the generating function by length for *long* fully commutative elements $\tilde{w} \in \tilde{S}_n^{FC}$ is

$$\sum q^{\ell(\tilde{w})} = \frac{q^n}{1-q^n} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q^2.$$

Corollary. The coefficients of $f_n(q)$ are eventually periodic with period dividing n. **When n is prime**, the period is 1, and $a_i = \frac{1}{n}\left(\binom{2n}{n} - 2\right)$.

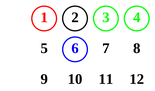
Proof. For i sufficiently large, all elements of length i are long. The generating function is simply some polynomial over $(1-q^n)$:

$$\sum_{\text{long } \tilde{w}} q^{\ell(\tilde{w})} = \frac{q^n}{1-q^n} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 = \frac{P(q)}{1-q^n} = P(q)(1 + q^n + q^{2n} + \cdots)$$

When n is prime, an extra factor of $(1 + \cdots + q^{n-1})$ cancels, giving $\frac{P'(q)}{1-q}$.

Short elements are hard

For short elements $\tilde{w} \in \tilde{S}_n$, the base window for w^0 is $[a, \ldots, a, b, \ldots, b, c, \ldots, c]$, and there is more interaction:



No a can invert with an a or b. No c can invert with a b or c.

- Count $\tilde{w}$ where some a intertwines with some c.
- Count $\tilde{w}$ w/o intertwining and 0 descents in the b's.
- Count $\tilde{w}$ w/o intertwining and 1 descent in the b's.
  - Not so hard to determine the acceptable finite permutations w.
  - Result: Contributions such as $\sum_{M \geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left(\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q$.
- Count $\tilde{w}$ w/o intertwining and 2 descents in the b's.
- Count $\tilde{w}$ which are finite permutations. (Barcucci et al.)
  - Use methods of Bousquet-Mélou to solve functional recurrences, such as $D(x,q,z,s) = N(x,q,z,s) + \frac{xqs}{1-qs}\big(D(x,q,z,1) - D(x,q,z,qs)\big) + xsD(x,q,z,s)$

References

Hanusa, Jones. The enumeration of fully commutative affine permutations *European Journal of Combinatorics.* Vol 31, 1342–1359. (2010)

Hanusa, Jones. Abacus models for parabolic quotients of affine Coxeter groups [*Generalizes combinatorial models to $\tilde{B}$, $\tilde{C}$, $\tilde{D}$*] *arXiv:1105.5333*

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