Notes on the average size of an s,t-core partition

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1. CH, 27 July 2011

The formula I found for the average number of boxes in a $(3,3m+1)$ core partition is $(5m+3m^2)/4$; does this agree with your formula Drew?

Here is method I used to find this formula: [I realize that some of my conventions for labels are non-standard; sorry.]

- I determined which alcoves are active for m. (See Figure 1.) If $\alpha_1 = \{\sqrt{3}/2, -1/2\}$ and $\alpha_2 = \{0,1\}$, then the root lattice coordinates ($f(x,y) = y\alpha_1 + x\alpha_2$) for which there is an active alcove satisfy the three inequalities

$$x + y \leq m+1, -x+2y \geq -m, \text{ and } 2x - y \geq -m.$$

  The attached alcove is the one that is closest to the center of the fundamental alcove.

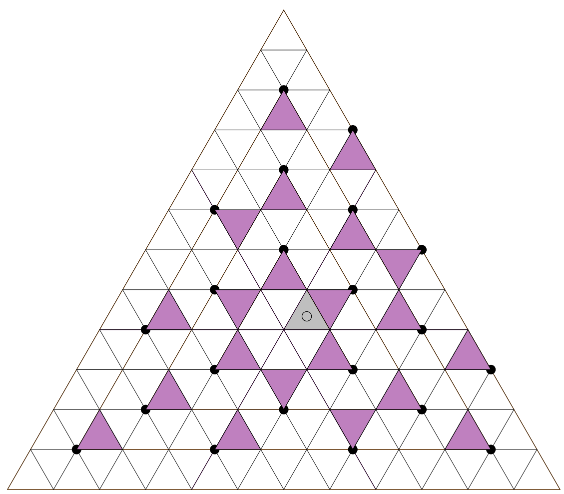

Figure 1. Active alcoves

- I then determined the number of parallel hyperplanes ($\#hyp$) that separate the alcove from the fundamental alcove, repeating for each family of parallel hyperplanes (H_0,H_1,H_2). Using the same convention ($y\alpha_1 + x\alpha_2$), there is a simple formula for this number. (See Figure 2.)
  - For H_0: $f[\#].(\alpha_1+\alpha_2) \leq 0$, then $\#hyp = x+y$, otherwise $\#hyp = x+y-1$.
  - For H_1: $f[\#].(\alpha_1) \leq 0$, then $\#hyp = -x+2y$, otherwise $\#hyp = -x+2y-1$.
  - For H_2: $f[\#].(\alpha_2) \leq 0$, then $\#hyp = 2x-y$, otherwise $\#hyp = 2x-y-1$.
- Collecting the $\#hyp$ statistics from all three families of parallel hyperplanes gave the biggest surprise. (See Figure 3.)

  It appears that for i from 0 to m, there are a constant number of alcoves with statistic $\#hyp = i$, with this constant number equal to $4m+3$. For i from $m+1$ to $2m$, the number of alcoves with statistic $\#hyp = i$ exactly equals $2m+1-i$. I am optimistic that we could figure out how to generalize this.

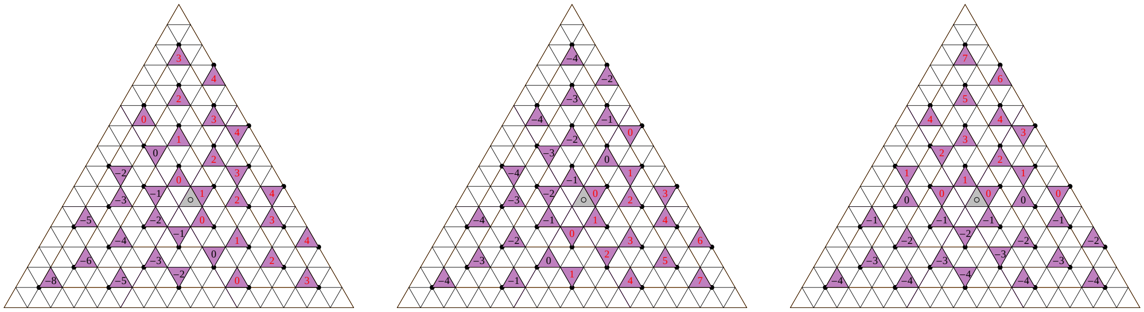

FIGURE 2. Number of hyperplanes separating active alcoves from fundamental alcove

0																				
3																				
0	1	2																		
7	7	1																		
0	1	2	3	4																
11	11	11	2	1																
0	1	2	3	4	5	6														
15	15	15	15	3	2	1														
0	1	2	3	4	5	6	7	8												
19	19	19	19	19	4	3	2	1												
0	1	2	3	4	5	6	7	8	9	10										
23	23	23	23	23	23	5	4	3	2	1										
0	1	2	3	4	5	6	7	8	9	10	11	12								
27	27	27	27	27	27	27	6	5	4	3	2	1								
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14						
31	31	31	31	31	31	31	31	7	6	5	4	3	2	1						
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16				
35	35	35	35	35	35	35	35	35	8	7	6	5	4	3	2	1				
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		
39	39	39	39	39	39	39	39	39	39	9	8	7	6	5	4	3	2	1		
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
43	43	43	43	43	43	43	43	43	43	43	10	9	8	7	6	5	4	3	2	1

FIGURE 3. Table of $\#hyp$ statistics

- Now we can calculate the average size of a $(3, 3m+1)$ simultaneous core partition:

(1)
$$\left((4m+3)\sum_{i=1}^{m}\binom{i+1}{2} + \sum_{i=m+1}^{2m}\left[(2m+1-i)\binom{i+1}{2}\right]\right)\Big/\left(\frac{2+5m+3m^2}{2}\right) = \frac{5m+3m^2}{4}$$

This is different from the method I used in Reykjavik, in which I determined a pattern for the number of alcoves in each diagonal, and used it to find a formula which included many floor functions. In both of my methods, I have not proved the formula for the number of alcoves separated by $\#hyp$ from the fundamental alcove. This will need to be justified.

2. DA, 12 August 2011

Your formula does indeed agree with my conjecture that the average size of an (n, p)-core is $(n+p+1)(n-1)(p-1)/24$. I like your method and I am optimistic that it can be extended to the general case.

Suppose $p = qn + r$, where q and r are the quotient and remainder. Say that the positive root $e_i - e_j$ with $i < j$ has "height" $j - i$. Let $H_{ij,k}$ denote the hyperplane defined by $(\cdot, e_i - e_j) = k$. Then we define the simplex $D^p(n)$ which is bounded by the set of hyperplanes

$$\{H_{ij,q+1} : j - i = n - r\} \cup \{H_{ij,-q} : j - i = r\}.$$

For example, $2 = 0 \cdot 5 + 2$, so the simplex $D^2(5)$ is bounded by the hyperplanes

$$H_{14,1}, H_{25,1}, H_{13,0}, H_{24,0}, H_{35,0}.$$

It is not too difficult (but tricky — as all this affine stuff is) to show that $D^p(n)$ contains p^{n-1} alcoves, and contains $\frac{1}{n+p}\binom{n+p}{n,p}$ elements of the root lattice, which are in bijection with (n,p)-cores. We can associate to each root vector the alcove touching it which is closest to the fundamental alcove. Call these the "root alcoves".

Now here's a conjecture based on your email: For each $i \geq 0$, let $f(n,p,i)$ be the sum, over positive roots α, of the number of root alcoves in $D^p(n)$ that are separated from the fundamental alcove by i hyperplanes perpendicular to α. Then the total number of boxes in the set of (n,p)-cores is the sum over i of $f(n,p,i)\binom{i+2}{2}$.

Based on your email, it is very possible that $f(n,p,i)$ has a formula simple enough such that the above sum simplifies to the desired formula.

If the general formula $f(n,p,i)$ is as simple as the $p = mn+1$ case, then we can probably guess it from a small amount of experimental data. I don't yet have an idea to prove such a formula.

3. DA, 12 August 2011, Part II

Here's an interpretation of $f(n, mn+1, 0)$, which I can prove.

Consider the lattice paths from $(0,0)$ to (n, mn) staying weakly above the diagonal. Say that a box above the path is in column i if it is between vertical lines $y = i-1$ and $y = i$. Then $f(n, mn+1, 0)$ is the sum over paths of number of boxes below the path (but above the diagonal) in columns with coordinate divisible by m.

I do not know a closed formula, but presumably this interpretation will help to find one.

Let lattice paths from $(0,0)$ to (n, mn) staying weakly above the diagonal be called m-Fuß-Catalan paths. We will consider boxes below the paths and above the diagonal. Then for $1 \leq i \leq m-1$ we have

$$f(n, mn+1, i) = \sum \big[(\#\text{ boxes below path in columns of residue } i \mod m) - (\#\text{ boxes below path in columns of residue } i-1 \mod m)\big],$$

where the sum is over m-Fuß-Catalan paths.

4. CH, 15 August 2011

I had a few questions about what you wrote, especially about the $f(n,p,i)$ numbers. You define them first in terms of root alcoves, and then in terms of lattice paths. You mention that you can prove the interpretation of $f(n, mn+1, 0)$, and then you sent a formula for $f(n, mn+1, i)$ with a similar interpretation.

(1) Does this mean that you have proved the correspondence between the root alcove interpretation and the lattice path interpretation for $f(n, mn+1, i)$?
(2) Is the bijection easy to see?

If I am understanding you correctly, you are implying that this bijection is another way to get a handle on the quantity $f(n, mn+1, i)$, but that even with this alternate combinatorial interpretation, you are unsure of how to use these interpretations to find a formula for $f(n, mn+1, i)$.

(3) Is this a fair statement?

(4) Would you say that a goal for this project would be to find a formula for $f(n, mn+1, i)$?
(5) How are these numbers $f(n, mn+1, i)$ related to the (q,t)-Fuß-Catalan numbers that I have seen in your work and in the work of Christian Stump?

Here are more general questions:

(6) Why are the elements of the root lattice in $D^p(n)$ in bijection with (n,p)-cores? I am under the impression that it has something to do with left versus right action of generators, but I am unclear on why these (n,p)-cores should necessarily be contained in $D^p(n)$. Is there some reference to read about this?
(7) Where does this simplex $D^p(n)$ come from?

5. DA, 15 August 2011

It'll take me a few days to get back to you in detail, but the answer to most of your questions is yes. I think I can prove the lattice path interpretation of $f(n, mn+1, i)$, up to $i = m-1$ (it might need to be tweaked, but something like it should be true). Beyond that I don't know how to think about it yet. I don't know a closed formula for this, but I am guessing that $f(n, mn+1, i)$ is constant in this range, and then decreases in some simple way for larger i.

Yes, it seems that the goal of the project is to guess and prove a formula for $f(n, p, i)$, and then use this to analyze the distribution of (n,p)-cores, in particular to prove the conjecture on the average size.

I think this is the standard bijection from the root lattice to n-cores: Given a root vector $(r_1, \ldots, r_n)$, fill in the n-abacus with the highest bead on the i-th runner at level r_i. The resulting abacus determines the beta set of an n-core. To say that this n-core is also a p-core (for p coprime to n) means that if you substract p from any bead label, the resulting labeled position must also contain a bead. This condition can be alternatively stated as a set of linear inequalities on the coordinates of the root; exactly those that define the simplex $D^p(n)$.

6. CH, 13 September 2011

With help from your responses to previous emails, I am now feeling more comfortable with the structure of $D^p(n)$. After doing some programming in Mathematica, I also feel more comfortable with the lattice paths, and Mathematica will now generate for me many pictures and data related to lattice paths (see attached). I would appreciate it if you could write down the bijection you state for $f(n, mn+1, 0)$ between root alcoves and lattice paths (or send me an appropriate reference). I am especially surprised that these multidimensional alcove pictures condense down to a two-dimensional lattice path interpretation (!).

My initial (naive) investigations into the bijection and into your lattice path definition of $f(n, mn+1, i)$ makes me wonder if your formula should be

$$f(n, mn+1, i) = \sum \big[(\#\text{ boxes below path in columns of residue } (i+1) \mod m) - \\ (\#\text{ boxes below path in columns of residue } i \mod m)\big],$$

instead of

$$f(n, mn+1, i) = \sum \big[(\#\text{ boxes below path in columns of residue } i \mod m) -$$
$$(\#\text{ boxes below path in columns of residue } i-1 \mod m)\big],$$

Alternatively, it may be the case that residues are shifted by 1 (and that residues should start in the first column with zero), or perhaps that I am completely misunderstanding the situation. The reason for my concern is that it appears that the sum (boxes of residue 1)−(boxes of residue 0) is negative in the current terminology. In addition, when I run the numbers, it does not seem to work out quite right. For example, from the attached figures we should be able to calculate $f(3, 7, 1)$ and $f(3, 10, 1)$. The number of boxes in each residue class is calculated to the right of each picture (please verify that I have your notion of residue correct). From this, when we sum over residue classes in all paths we have (class 1: 12, class 2: 21) when $m = 2$, and (class 1: 18, class 2: 30, class 3: 42) when $m = 3$. As you can see when $m = 3$, the differences between residue classes are indeed constant (this holds true for $m = 4$ through $m = 10$ as well), however the numbers are not quite right. I think that the definition can not be too far off, but something is not right.

7. DA, 13 September 2011

Here I'm using a bijection which I think appears in Fishel and Vazirani's paper, but I'm not sure. I'll describe it. Think of a square grid and for $1 \leq i < j \leq n$ label the box with top-right corner (i, j) with the root $e_i - e_j$. Given a dominant region of the Shi arrangement, the hyperplanes $e_i - e_j = 1$ **below** the region correspond to the boxes **above** a Dyck path from $(0, 0)$ to (n, n), and this is a bijection. (This is well known). Now consider the m-Shi arrangement with hyperplanes $H_{e_i-e_j,k}$ for $-m+1 \leq k \leq m$. Let R be a dominant region in this arrangement. Then for each $1 \leq k \leq m$ we get a Dyck path $(0, 0)$ to (n, n) corresponding to the hyperplanes $H_{e_i-e_j,k}$ below R. Add these Dyck paths (that is, add the shapes above them row by row) to get a path from $(0, 0)$ to (mn, n) staying above the diagonal. Claim (and this — I think Brant told me — is in some paper by Fishel and Vazirani): This is a bijection from dominant m-Shi regions to Fuss-Catalan paths $(0, 0)$ to (mn, n).

I think the proof involves placing the hyperplanes $H_{e_i-e_j,k}$ in the boxes with column residue k. Then one can translate information about separating (resp. nonseparating) hyperplanes with a given height into information about boxes above (resp. below) the lattice path with a given column residue. This is the idea I used — and as you mention I may have got the conventions slightly off.

8. CH, 21 September 2011

Here is an update of a number of different ideas I've thought about in the past week.

8.1. **Lattice Path Interpretation.** I have come to believe that the Fuss-Catalan path interpretation is not the right approach to this problem.

One reason is because of the problems with the idea of residues that we would want to be true. This goes back to your method of constructing the m-Dyck paths by combining m independent Dyck paths. Consider the case in A_2 with $m = 3$, shown in Figures 4 and 5. In Figure 4, you can see the gluing of the three independent Dyck paths (one blue, one green,

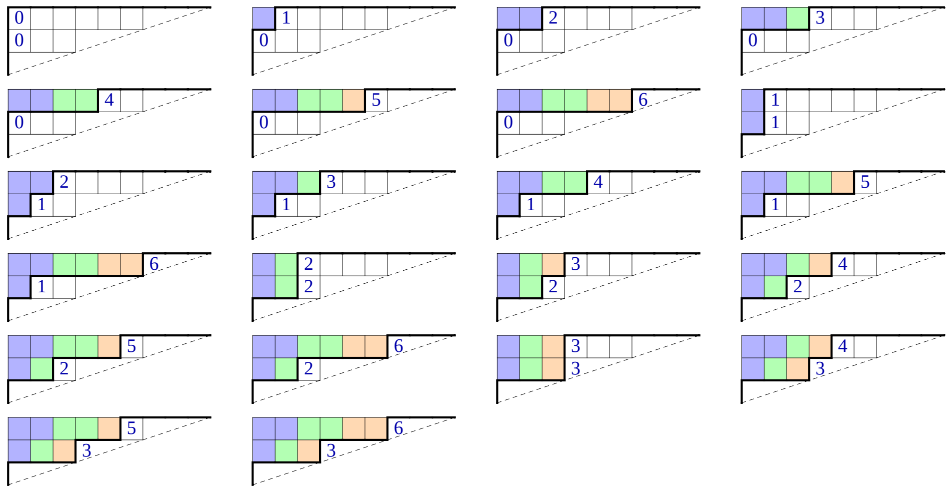

FIGURE 4. The 22 lattice paths from $(0,0)$ to $(9,3)$, with their corresponding partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_1 \leq 6$, $\lambda_2 \leq 3$, and $\lambda_3 \leq 0$. The coloring of the boxes corresponds to the bijection with Figure 5.

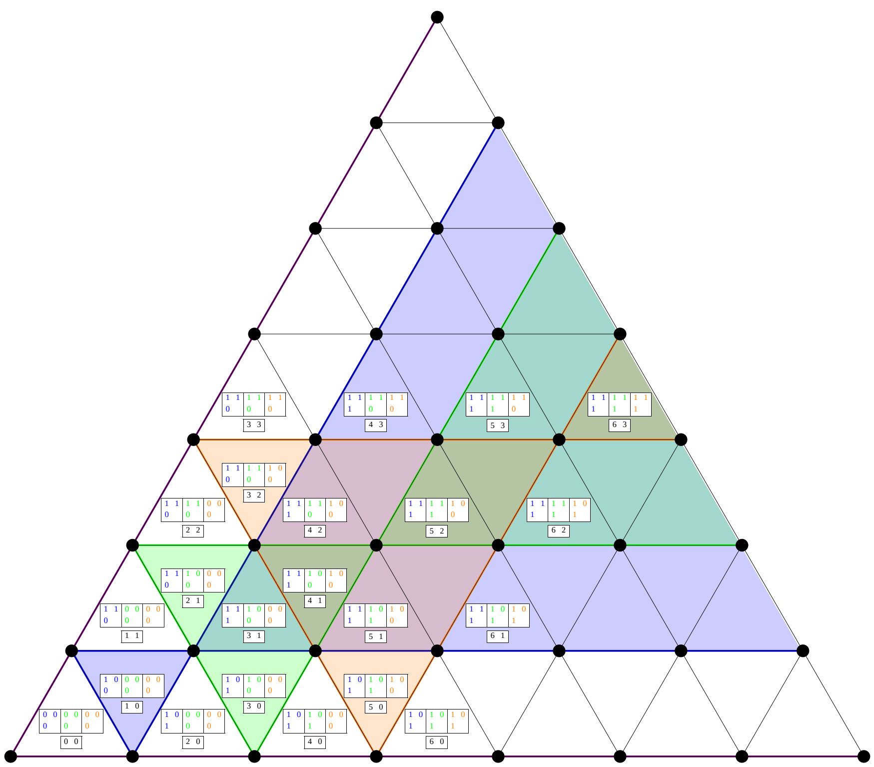

FIGURE 5. The 22 3-minimal alcoves in the 3-Shi arrangement of A_2. The numbers correspond to the placement of the alcoves with respect to the hyperplanes $H_{\alpha,1}$ (in blue), $H_{\alpha,2}$ (in green), and $H_{\alpha,3}$ (in orange). Adding the numbers of all colors in column 1 gives the first number in the pair below, and adding the numbers in column 2 gives the second number in the pair below. This pair gives the correspondence with the lattice paths and partitions in Figure 4.

one orange), which corresponds to the positions of each alcove in Figure 5 with respect to the hyperplanes $H_{\alpha,k}$ for $k = 1$ (blue), $k = 2$ (green), and $k = 3$ (orange).

The residues that should be placed in the boxes above the lattice path must be $\{0, 1, 2\}$, depending on the hyperplane traversed, $\{\theta, \alpha_1, \alpha_2\}$ respectively. We notice that a 0 corresponds to the first occurrence of a colored box in the first row, a 1 corresponds to the first occurrence of a colored box in the second row, and a 2 corresponds to the second occurrence of a colored box in the first row. In this way, it is possible to read off the number of hyperplanes of each type* that separate the alcove from the fundamental alcove, **as long as** we know the coloring scheme for the boxes above the lattice path.

However, even this is not clear cut because it is unclear how to resolve an m-Dyck path into m independent Dyck paths. There must be a rule that determines these m paths (as we do have a bijection), but this rule seems complicated. For example, consider the lattice path corresponding to partitions $(3, 2, 0)$ and $(4, 2, 0)$. Except for these two cases, we might conjecture that we use as many colors as there are boxes in the second row of the partition. This rule does not work in these two cases because it is impossible to have only blue and green boxes—if the alcove is separated from the fundamental alcove by hyperplanes $H_{\theta,1}$, $H_{\alpha_1,1}$, $H_{\alpha_2,1}$ and $H_{\theta,2}$, then it must be the case that the alcove is separated from the fundamental alcove by $H_{\theta,3}$ as well.

The more compelling reason why I am not happy with the lattice path method has to do with the (*) above, because it is **not** the case that we are keeping track of **all** hyperplanes separating particular alcoves from the fundamental alcove. For instance, consider the alcove corresponding to partition $(6, 3, 0)$. This alcove is separated from the fundamental alcove by $H_{\alpha_1,1}$, $H_{\alpha_1,2}$, and $H_{\alpha_1,3}$ as well as $H_{\alpha_2,1}$, $H_{\alpha_2,2}$, and $H_{\alpha_2,3}$, and by $H_{\theta,1}$, $H_{\theta,2}$, and $H_{\theta,3}$. In addition, it is separated by $H_{\theta,4}$, $H_{\theta,5}$, and $H_{\theta,6}$, which are not taken into account by the Dyck path interpretation (or I do not see how). We can see that this will lead to an overcount for $f(3, 3m + 1, i)$ for $i \leq m$ and an undercount of $f(3, 3m + 1, i)$ for $i > m$.

This leads into the discussion of the next topic, which I hope will serve as a general framework for solving this problem.

8.2. **Pyramids.** Fishel, Tzanaki, and Vazirani's [FTV09] discusses coordinate systems for alcoves and regions of the m-Shi hyperplane arrangement. The coordinates of an m-minimal alcove in the m-Shi arrangement can be represented by a table of numbers, $\{k_{ij}\}_{1 \leq i \leq j \leq n-1}$, where k_{ij} tells us how many hyperplanes perpendicular to $\alpha_{ij} = \varepsilon_i - \varepsilon_{j+1}$ separate the alcove from the fundamental alcove. See Figure 6 for the coordinates of every 4-minimal alcove in the 4-Shi arrangement.

Fishel et al. also use and define the coordinates of a region in the m-Shi arrangement by setting $e_{ij} = \min(k_{ij}, m)$. In fact, these are the coordinates that would be found by adding the colored arrays of numbers in Figure 5. Fishel et al. use the terminology *Shi tableau* to reference the staircase partition of the *Shi coordinates* of an alcove; I prefer the name *pyramid*, which comes from Richards's [Ric96].

What these Shi coordinates do is give us a new method to calculate $f(n, mn + 1, i)$, which will be exactly the number of occurrences of i in $\{k_{11}, k_{22}, \ldots, k_{n-1,n-1}, k_{1,n-1}\}$. Analyzing the structure of m-minimal alcoves in the 3-Shi arrangement now **does** prove that the average size of a $(3, 3m + 1)$ simultaneous core partition is $(5m + 3m^2)/4$.

Proposition 8.1. *The average size of a* $(3, 3m + 1)$ *simultaneous core partition is* $(5m + 3m^2)/4$.

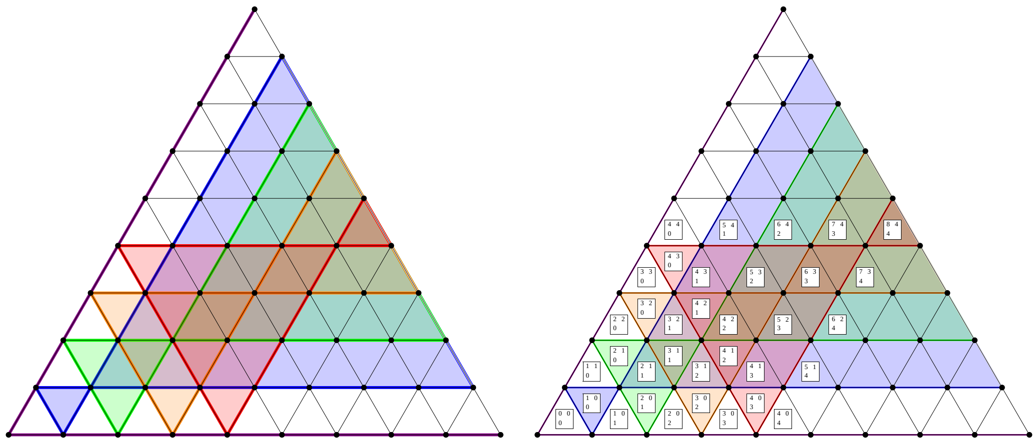

FIGURE 6. The 4-Shi arrangement in A_2 along with pyramid coordinates of every 4-minimal alcove.

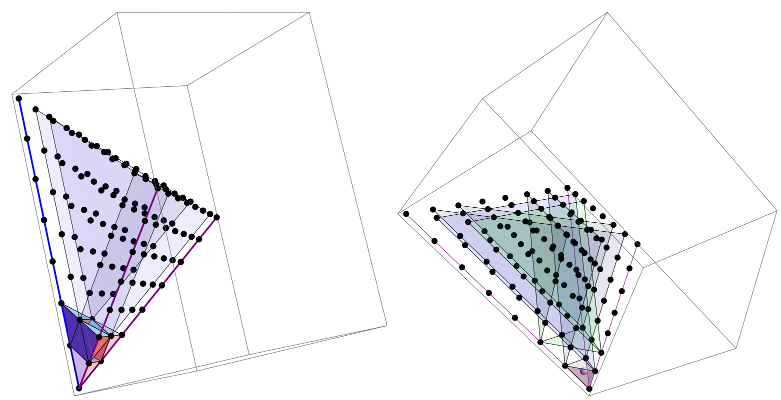

FIGURE 7. Alcoves in the 1-Shi and 2-Shi arrangements in A_3

Proof. When $n = 3$, we note that the Shi coordinates for m-minimal alcoves $\begin{bmatrix} k_{12} & k_{11} \\ k_{22} & \end{bmatrix}$ are either of the form $\begin{bmatrix} i+j & i \\ j & \end{bmatrix}$ for $0 \leq i, j \leq m$ or $\begin{bmatrix} i+j+1 & i \\ j & \end{bmatrix}$ for $0 \leq i, j \leq m-1$ and $i + j \leq m - 1$.

We first determine the number of occurrences of k for $0 \leq k \leq m$. The value k can occur as a value for i or j in the former case (in $2(m+1)$ ways) or as a value for i or j in the latter case (in $2(m-k)$ ways). k can also occur as an $i+j$ or $i+j+1$ in $2k+1$ ways, for a total of $4m+3$ ways. When $k \geq m+1$, there are $2m+1-k$ ways k can appear. We conclude that Equation 1 holds. □

In order to answer this question for other simultaneous core partitions, we will need to understand the structure of the m-minimal alcoves and their Shi coordinates. The characterization of valid Shi tableaux has already been done, but it seems less than obvious about how to use the characterization to count occurrences. As I mentioned in an email yesterday, I worked to implement alcoves in A_3 in *Mathematica*; results are in Figure 7. It was surprising to me how many alcoves are separated from the fundamental alcove by $H_{\theta,1}$ and not by $H_{\theta,2}$.

8.3. **Questions.**

Question 8.1. *You mention that the bijection between (boxes above a Dyck path) and (hyperplanes below a dominant region) is well known. Do you have a reference or two available?*

Question 8.2. *Perhaps Monica knows about the generalization to Fuss-Catalan paths and see if she has any ideas about an interpretation of the boxes above the paths.*

Question 8.3. *Do you have any thoughts on approaching higher-dimensional arrangements?*

9. CH, 27 September 2011

After asking Question 8.2 to Monica, she forwarded the message onto Susanna Fishel (`susannadf@gmail.com`), who replied with the following information.

> "Pak/Postnikov/Stanley's bijection from regions to parking function, when restricted to dominant regions, shows that the number of boxes in the partition fitting in a staircase (=lattice path) is the number of hyperplanes separating the region from the origin. I think the partitions in $((n-1)m, (n-2)m, ..., m)$ with part $i = (n-i)m$ correspond to regions with H_{α_i} as separating wall. I think."

9.1. **Direct counting of alcoves in A_3.**

Here is a guess about generalizing the counting method from A_2 to A_3. I am imagining that the set of m-minimal alcoves will look like a "solid" $m+1$-dilation of the fundamental alcove $\mathcal{A}_0$ with a "holey" $m+1$-dilation of the fundamental alcove reflected about the hyperplane $H_{\theta,m+1}$. In A_2, we see this structure in Figure 6, where all alcoves within the 5-dilation of $\mathcal{A}_0$ are 4-minimal alcoves and only certain alcoves in the reflected 5-dilation of $\mathcal{A}_0$ are m-minimal alcoves. Given a characterization of the m-minimal alcoves for higher dimensions (and a proof that the characterization is correct) will provide a way to verify the average simultaneous core formula for $(n, mn+1)$.

Question 9.1. *Is anything known about a nice alcove interpretation of (n,t)-core partitions when $t \neq mn+1$?*

9.2. **Abaci and lattice paths in the work of Anderson.**

When reading through Fishel et al.'s discussion of Shi tableaux, I read backward and saw that these Shi tableaux characterize the relative levels of the lowest beads in the abaci, which means that there should be a way to investigate these simultaneous core partitions using abacus diagrams instead of Shi tableaux. Figure 8 shows the abacus diagram that corresponds to alcoves in type A_2. The *normalized* abacus diagram is given, where the runners are rotated so that the first gap occurs in position 0.

Going back to the work of Anderson [And02], we see that there is another a lattice path interpretation for partitions that are n and $mn+1$ cores (in fact, for s and t cores in general).

Definition. Given relatively prime integers s and t, we will place certain integers on a rectangular lattice of dimensions $(0,s) \times (0,t)$. We place integer $st - xs - yt$ in position (x,y). Note that the integers below the line $l : y = -\frac{s}{t}x + s$ are positive. There is a bijection between lattice paths from $(0,s)$ to $(t,0)$ staying below the line l and abacus diagrams representing simultaneous (s,t) core partitions. Given such a lattice path, read off the right-most number in each row; this number is the position of the highest gap on a runner of the abacus. An example is given in Figure 9.

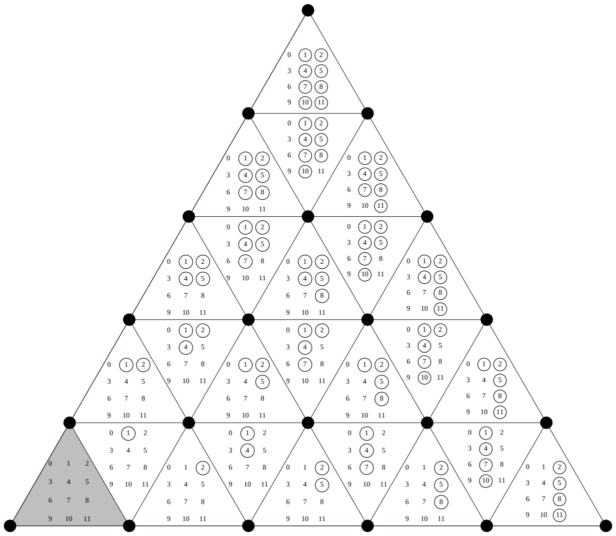

FIGURE 8. The abacus diagrams corresponding to the alcoves in type A_2.

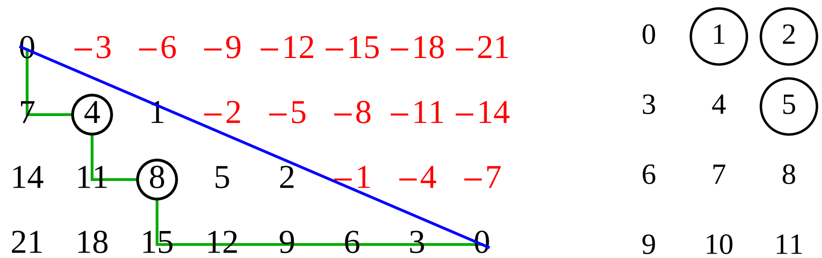

FIGURE 9. A lattice path from $(0,3)$ to $(7,0)$ staying below $3-\frac{3}{7}x$, along with the the corresponding abacus diagram for a simultaneous $(3,7)$-core partition.

Importance: From Anderson's characterization, we know the set of abaci that correspond to simultaneous core partitions. If we can convert the knowledge of the set of valid abaci to statistics on separating hyperplanes in alcoves (for the general case of s and t cores), and this will determine a formula for the size of an average core! Alternatively, we would be able to read off the size of the cores by counting bead-gap pairs in the abaci.

What is surprising is that Anderson's lattice path interpretation is **different** from the aforementioned lattice paths from $(0,0)$ to (mn,n). Any path from $(0,n)$ to $(mn+1,0)$ is also a path from $(0,n)$ to $(mn,0)$, so we might imagine that there is a simple bijection, but I have not yet found it.

Question 9.2. *Is there a bijection between Anderson's lattice path interpretation and the other lattice path interpretation when dealing with simultaneous n and $nm+1$ cores?*

10. DA, 27 SEPTEMBER 2011

10.1. Answers to Chris's Questions.

Answer to Question 8.1. In my thesis I attributed this to Cellini and Papi, "ad-nilpotent ideals of a Borel subalgebra II". I am sure this was known to Shi in the 1980s but I don't have an exact reference. (Shi wrote so many papers with similar content that I have a hard time pinpointing his results.)

Answer to Question 9.1. Yes. Susanna and Monica wrote a paper proving that the $(n, nm-1)$-cores correspond to the **maximal** alcoves in the bounded chambers of the m-Shi arrangement. (Citation: [FV09])

Answer to Question 9.2.

I guess that the bijection described in Section 2.5 of Nick Loehr's thesis will work. `http://www.combinatorics.org/Volume_12/PDF/v12i1r9.pdf`

My guess is based on the following: I believe that Nick's bijection is a generalization of the zeta map that exchanges (area,dinv) with (bounce,area) for $m = 1$. This is Theorem 3.15 (page 50) in Jim's book. `http://www.math.upenn.edu/~jhaglund/books/qtcat.pdf`

And the zeta map is the $m = 1$ version of the bijection you want. (I can prove this, but I don't think it's written down anywhere.)

10.2. **Bouncy bijections.** I'll tell you how I understand the (area,dinv)↔(bounce,area) bijection without going through the alcove picture.

Start with a lattice path and label the boxes to the right of the up-steps as follows: Put 1 in the bottom left box, then put 2 in the next box you see along a slope 1 diagonal. Continue along the same diagonal to place 3, etc until you run out of boxes. Then continue with the next higher slope 1 diagonal, etc. This is called the diagonal filling.

Now, to each vertical domino of boxes filled with $i < j$ we associate the root $e_i - e_j$. This defines a bijection to antichains in the root poset. Go from an antichain to a new lattice path by indexing a box by (a, b) if its top-right corner is (a, b). Then the shape above the desired lattice path corresponds to the boxes in the filter generated by the antichain. The antichain elements are the boxes sitting in the valleys of the path.

So essentially the pairs of two consecutive up-steps in the original path go to the valleys of the new path. I find this much easier to compute than the definition in Jim's book.

11. CH, 10 October 2011

11.1. **Recent Thoughts.**

I learned about the set of co-filtered chain of ideals that are in bijection with the alcoves. As the alcove coordinates can be directly read from such a chain, it is expected that these chains have as difficult a structure as the alcove coordinate descriptions. I imagine that there is some action of the generators of W on these chains; perhaps that has been discussed somewhere in the literature.

I have perused the literature related to the (area,dinv) ↔ (bounce,area) correspondence, including the rule for finding "bounce" in m-Dyck paths (some references suggested by Drew, thanks). The construction is quite complicated. Drew mentions that this should be the rule that creates the bijection with Anderson's lattice path, but I am not seeing the correspondence at all; I do not currently see any reason why they should be the same. I plan to work some examples.

It is clear that $(n, mn+1)$-core partitions have a Fuss-Catalan lattice path interpretation, but it is unclear that (n, p)-core partitions have a Fuss-Catalan lattice path interpretation. However, Anderson's lattice path interpretation does work for (n, p). In addition, looking at Anderson's lattice paths, it is natural that $(n, mn + 1)$ and $(n, mn - 1)$ are nice cases, and other cases are not as nice because the order of the residues on the levels of the lattice paths are irregular.

I like your diagonal filling method to compute the corresponding antichain. If I work with it further, I suppose I will come to understand how to derive it from its other definition.

11.2. **Generating more data.**

Using Anderson's lattice path definition, I was able to generate some more data toward Drew's conjecture! There is some good news, but mostly some bad news. The bad news is that Drew's conjecture does not work for $n > 3$. The good news is that with more data we could possibly understand how to prove a formula in general. In addition, it appears that Drew's conjecture holds for $n = 3$ and **all** $p \equiv 1, 2 \mod 3$; it remains to prove for $p \equiv 2 \mod 3$ since $p \equiv 1 \mod 3$ was proved in Proposition 8.1.

I am able to generate data for all relatively prime pairs of numbers (n, p). And in fact, there appears to be plenty of similarities between the data for different residues mod n, but the data for 1 modulo n is the nicest. Here is the data for $n = 4$ and $n = 5$:

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
5	29	21	5	1																		
9	80	60	52	14	10	3	1															
13	157	125	105	97	27	23	16	6	3	1												
17	260	216	184	164	156	44	40	33	23	10	6	3	1									
21	389	333	289	257	237	229	65	61	54	44	31	15	10	6	3	1						
25	544	476	420	376	344	324	316	90	86	79	69	56	40	21	15	10	6	3	1			
29	725	645	577	521	477	445	425	417	119	115	108	98	85	69	50	28	21	15	10	6	3	1

FIGURE 10. Table of values for $(4, p)$-simultaneous core partitions.

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
6	121	63	18	7	1																
11	577	349	247	72	66	33	16	4	1												
16	1609	1109	783	631	180	185	157	90	57	29	10	4	1								
21	3453	2579	1927	1497	1289	360	382	365	303	190	136	87	47	20	10	4	1				
26	6345	4995	3915	3105	2565	2295	630	675	675	624	516	345	265	190	124	71	35	20	10	4	1

FIGURE 11. Table of values for $(5, p)$-simultaneous core partitions.

We can see some patterns emerging. For example, notice that when $n = 3$, the number of regions separated by k hyperplanes broke into two clearly delimited parts. In the case for $n = 4$, there are three parts, and when $n = 5$, there are four parts. Also, we see that the last part consists of triangular (tetrahedral) numbers for $n = 4$ ($n = 5$). When $n = 4$, both the red and green parts have entries that are predictable; the differences between consecutive terms are $\ldots, 80, 68, 56, 44, 32, 20, 8$ and $4, 7, 10, 13, 16, 19, \ldots$, which implies that the entries in both the red and the green parts satisfy a quadratic equation. For $n = 5$, the entries in the red parts satisfy a quadratic equation, while the other parts satisfy cubic equations.

We can conjecture a formula for $ac(4, 4m + 1)$:

Conjecture 11.1. *The average size of a* $(4, 4m+1)$ *simultaneous core partition is*

$$ac(4,4m+1) = \frac{\left(\begin{array}{c} \sum_{k=1}^{m} \left(4+12m+13m^2+6k^2-2k(1+6m)\right)\binom{k+1}{2}+ \\ \sum_{k=m+1}^{2m} \left(\frac{1}{2}(2+k-3k^2+5m+6km+m^2)\right)\binom{k+1}{2}+ \\ \sum_{k=2m+1}^{3m} \binom{3m+2-k}{2}\binom{k+1}{2} \end{array} \right)}{\binom{4m+5}{4}\Big/(4m+5)}$$

$$= \frac{1}{12}m(4m+5)(20+106m+193m^2+146m^3+39m^4)\Big/\binom{4m+5}{4}$$

$$= \frac{m(20+86m+107m^2+39m^3)}{4(1+2m)(3+4m)} \qquad (2)$$

A conjecture for $ac(4, 4m+3)$ should look similar, but it is not equal, so it is not the case that for $n = 4$ (and probably higher values of n), $ac(n, p)$ is a function of n and p. It feels that this data implies that the method of proof will not be uniform for all n; if there is a "nice" formula, it will likely involve some sort of reference to the structure of lattice paths or some other combinatorial interpretation. Perhaps the best we could hope for is a nice result for $(3, p)$ cores, and in a stretch both $(n, mn+1)$ and $(n, mn-1)$ cores? Actually I think that $(n, mn+1)$-cores and (n, p)-cores are of the same difficulty (quite hard), so if a formula is found for $ac(n, mn+1)$, then a formula for $ac(n, p)$ should be available with a little more nudging.

12. DA, 10 October 2011

I'm skeptical. I had quite a bit of data to support the original conjecture. Data with a and b into the teens. The same conjecture also holds when we restrict to self-conjugate (a, b)-cores. Here's a maple worksheet to compute the average size of an (a, b)-core. It currently displays the example $(a, b) = (6, 7)$, which works. It needs Stembridge's posets package. The algorithm is based on Anderson's bijection. Betas refers to the set of first-column hook-lengths. Alphas refers to the usual notation for an integer partition.

13. CH, 11 October 2011

Thankfully Drew is around to enlighten me to see the errors of my ways. I had made an incorrect assumption, that to calculate the number of boxes in the core partition, I would only need to tally the h_{ii} and h_{1n} terms from the Shi coordinates; what I really needed to tally was the values of *all entries* of the Shi coordinates. I modified my Mathematica code to do this and generate new tables similar to those above. (See Figures 12 and 13.)

We notice that values still break down into (colored) parts, each of which satisfies a polynomial of order $n-2$.

It is important to note that I am only giving the values for simultaneous $(4, 4m+1)$ and $(5, 5m+1)$ cores, which are nice families of cores and we can see that the first (red) part of the data is constant. This is not the case in general, such as in the case of $(5, 5m+2)$ core partitions. This makes me reconsider my claim that (n, p) in general will be of the same order of difficulty as $(n, mn+1)$ and $(n, mn-1)$.

We can update Conjecture 11.1, which (now) agrees with Drew's formula of $\frac{1}{24}(n+p+1)(n-1)(p-1)$:

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
5	37	37	9	1																		
9	94	94	94	28	16	3	1															
13	177	177	177	177	57	41	24	6	3	1												
17	286	286	286	286	286	96	76	55	33	10	6	3	1									
21	421	421	421	421	421	421	145	121	96	70	43	15	10	6	3	1						
25	582	582	582	582	582	582	582	204	176	147	117	86	54	21	15	10	6	3	1			
29	769	769	769	769	769	769	769	769	273	241	208	174	139	103	66	28	21	15	10	6	3	1

FIGURE 12. Corrected table of values for $(4, p)$-simultaneous core partitions.

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
6	176	176	56	11	1																
11	733	733	733	278	173	53	22	4	1												
16	1917	1917	1917	1917	777	587	377	146	83	37	10	4	1								
21	3964	3964	3964	3964	3964	1664	1364	1039	688	310	204	119	57	20	10	4	1				
26	7110	7110	7110	7110	7110	7110	3050	2615	2150	1654	1126	565	405	270	162	83	35	20	10	4	1

FIGURE 13. Corrected table of values for $(5, p)$-simultaneous core partitions.

Conjecture 13.1. *The average size of a* $(4, 4m+1)$ *simultaneous core partition is*

$$
\begin{aligned}
ac(4,4m+1) &= \frac{\left(\begin{array}{l}(6+18m+13m^2)\sum_{k=1}^{m}\binom{k+1}{2}+ \\ \quad \sum_{k=m+1}^{2m}\left(\frac{1}{2}(6-k^2+21m+17m^2-k(5+6m))\right)\binom{k+1}{2}+ \\ \quad \sum_{k=2m+1}^{3m}\binom{3m+2-k}{2}\binom{k+1}{2}\end{array}\right)}{\binom{4m+5}{4}/(4m+5)} \\
&= \frac{1}{3}m(1+m)(1+2m)(3+2m)(3+4m)(4m+5)\Big/\binom{4m+5}{4} \\
&= m(3+2m) \qquad (3)
\end{aligned}
$$

Drew also mentions self-conjugate simultaneous cores. One would hope that there would be some conceptual reason why the average size of the self-conjugate simultaneous core gives the same formula for the average size of all the simultaneous cores.

Question 13.1. *What is the conceptual reason behind the equivalence of* $ac(n,p)$ *and* $asc(n,p)$*? Would an involution argument work? This may bring the focus "squarely" on the self-conjugate simultaneous cores.*

Question 13.2. *From work of Fayers* [Fay11]*, we know that every simultaneous core partition is a subdiagram of the simultaneous core partition of maximal size. Is there a way to use this fact? I think there are some complications in that multiple copies of the same subdiagram may exist within the same maximal core.*

14. CH, 12 October 2011

We should be able to prove Conjecture 13.1 by using the same method as that to prove Proposition 8.1.

Idea for how to prove Conjecture 13.1. The six possible sets of pyramid coordinates for minimal alcoves in the m-Shi arrangement for A_3 are: (cases 0 through 5)

$$\begin{bmatrix} i+j+k & i+j & i \\ j+k & j & \\ k & & \end{bmatrix}, \begin{bmatrix} i+j+k+1 & i+j & i \\ j+k & j & \\ k & & \end{bmatrix}, \begin{bmatrix} i+j+k+1 & i+j+1 & i \\ j+k & j & \\ k & & \end{bmatrix}$$
$$\begin{bmatrix} i+j+k+1 & i+j & i \\ j+k+1 & j & \\ k & & \end{bmatrix}, \begin{bmatrix} i+j+k+1 & i+j+1 & i \\ j+k+1 & j & \\ k & & \end{bmatrix}, \begin{bmatrix} i+j+k+2 & i+j+1 & i \\ j+k+1 & j & \\ k & & \end{bmatrix}.$$

The restrictions on the values that (i,j,k) can take on in each case are dictated by the fact that we are not allowed to have two alcoves that are in the same region, that is, the region coordinates must be distinct. The way to ensure this is through the following restrictions.

Case	Set of restrictions on (i,j,k)
Case 0	no restrictions
Case 1	$i+j+k<m$
Case 2	$i+j<m$
Case 3	$j+k<m$
Case 4	$i+j<m$ and $j+k<m$
Case 5	$i+j<m$ and $j+k<m$ and $i+j+k+1<m$

We now can calculate the number of appearances of 0 up to $3m$ as coordinates in each case using generating functions.

In case 0, the values that $i+j+k$ can take on are the coefficients of the generating function $(1+x+\cdots+x^m)^3$. The values that $i+j$ or $j+k$ can take on are the coefficients of $(m+1)(1+x+\cdots+x^m)^2$, where the $m+1$ arises because of the m choices for the variable that is not involved. The generating function for the values that i or j or k can take on are $(m+1)^2(1+x+\cdots+x^m)$. We therefore have the generating function

$$f_0(x) = (1+x+\cdots+x^m)^3 + 2(m+1)(1+x+\cdots+x^m)^2 + 3(m+1)^2(1+x+\cdots+x^m).$$

Then we run into some trouble. First, it is not pleasant to get a formula for $[x^k]f_0$, because there is no simple form even for $[x^k]\big((1+x+\cdots+x^m)^3\big)$, which is OEIS sequence A109439. There was an interesting note in sequence number A027907; it implied that the coefficient of $[x^k]\big((1+x+\cdots+x^m)^n\big)$ equals the number of lattice paths from $(0,0)$ to (n,k) using steps of size $(1,0),(1,1),\ldots,(1,m)$. This brings us another appearance of lattice paths from $(0,0)$ to (n,mn).

There is another problem when we try to find a generating function approach for other cases. It is no longer the case that the possible values for $i+j+k$ is a product of three generating functions; the relationship is more complicated; much more so than the two variable case. There **must** be a better way to think about this since the numbers are so nice. □

I think it is clear that this is not feasible to do for all (n, p) simultaneous cores, and not even for all $(n, nm + 1)$ simultaneous cores. This also does not get at the reason why we should expect the number of appearances of 0 through m to be constant.

I generated the data for self-conjugate simultaneous cores and the data looks even more mysterious than the data for all simultaneous cores. I was hoping that it would be enlightening...

Up until now I have been generating data by first finding the valid abaci that represent the simultaneous cores, from which we can read off the coordinates of an alcove by calculating the number of levels between lowest beads on the abacus. The coordinates give the number of hyperplanes of type α that separate each alcove/region from the fundamental alcove, from which we can calculate the number of boxes in the core partition by recognizing that being separated by k hyperplanes corresponds to a total of $\binom{k+1}{2}$ boxes being added.

Perhaps there is a direct way to collect data on the size of each core from the abacus instead of passing through the alcoves? Of course there is—it has to do with the number of bead-gap pairs. Given highest gaps $(g_0, g_1, \ldots, g_{n-1})$ (ordered in increasing order), we can determine the number of boxes in the corresponding core partition by considering the number of bead-gap pairs generated by runners r_i and r_j for all $i < j$. And if we do this we find the exact same formula we did by passing through alcoves. Can we count another way? Or is there a way to count them in aggregate without resorting to counting each individual object?

15. CH, 14 October 2011

15.1. **Thoughts related to Sections 13 and 14.**

In order to prove a formula related to the data in the colored tables above, we will need to investigate the appearance of values of k in the pyramid coordinates in all cases. In the case that works to prove Proposition 8.1, the reason for the constancy is related to the way the number of appearances of k increase gradually for terms of the type $i + j$ or $i + j + 1$.

The $i+j$ contribution is $0, 1, 1, 2, 2, 2, 3, 3, 3, \ldots$, where the number of occurrences of $k \leq m$ is $k + 1$; similarly the contribution from entries of the form $i + j + 1$ is k. The contributions from the i and j coordinates are also easy to describe. In coordinates of the first type, the i and j each range independently from $0 \leq i, j \leq m$, so the appearances of k are equinumerous. In coordinates of the second type, i and j satisfy $i + j < m$, so the number of occurrences of k is $2(m - k)$. Since the increasing amounts of k for $i + j$ and $i + j + 1$ entries is cancelled out by the decreasing amounts of k for i and j entries, we have constancy.

Because of the data I generated, this appears to be a more general phenomenon; figuring out exactly what the red data is in Figures 12 and 13 would require understanding which sets add up to constants in this general case. That this is a more general phenomenon is quite surprising actually and might merit study on its own.

In addition, Case 0 is the only case in which $i + j + k$ can be larger than $m(n - 1)$, and therefore it makes sense that the black data are binomial coefficients.

But this is not what I wanted to think about today...

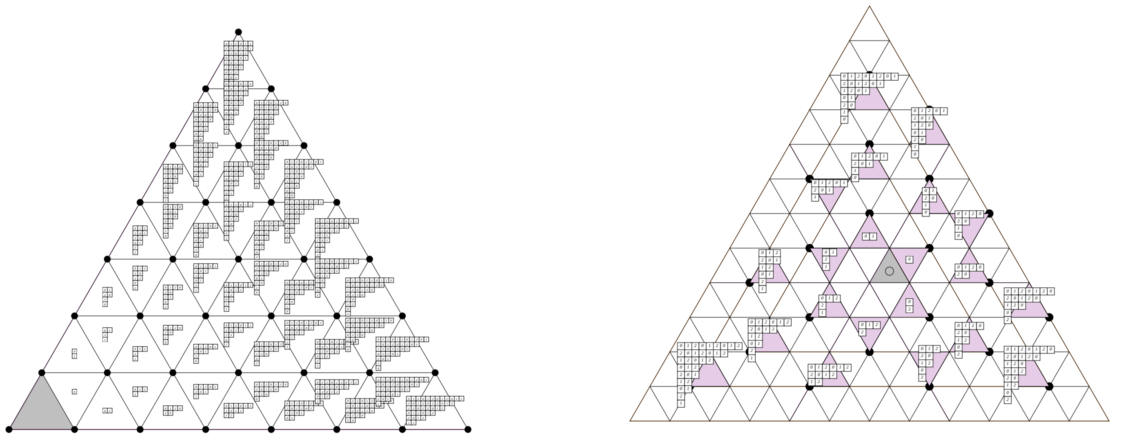

FIGURE 14. The core partitions that correspond to alcoves and inverse alcoves

15.2. What Ish means in normalized abaci.

Instead, I want take the abaci in alcoves picture (Figure 8) and cores in alcoves picture (Figure 14a) and turn them inside out to see what I could learn. (See Figures 15a and 14b, respectively.) I have found something beautiful that I suspect that Drew has already seen but perhaps it is something new to him as well. In any case, I have found a new way to approach the problem, which may be able to be generalized, but I do not know exactly how.

The basic idea is that the Ish statistic for alcoves in the inverse diagram is exactly the number of beads in the (normalized) abacus diagram, while the position of the lowest bead in the (normalized) abacus is determined by a modified Shi statistic. Since the abacus has two runners with beads, knowing these two pieces of information describes the abacus completely, from which we can determine the Shi coordinates of the alcove and therefore the statistics for how many i, j, and $i+j$. Since we know the coordinates of the alcoves in $D^p(n)$, and can determine a formula for the Ish statistic and the modified Shi statistic, then we can determine the formula for the number of appearances of k as a coordinate.

And I'm sure there are better formulas, but here are formulas for Ish and Shi as a function of a root lattice coordinate $\mathbf{x}$ in terms of the roots $\alpha_1 = (\sqrt{3}/2, -1/2)$ and $\alpha_2 = (0,1)$ and

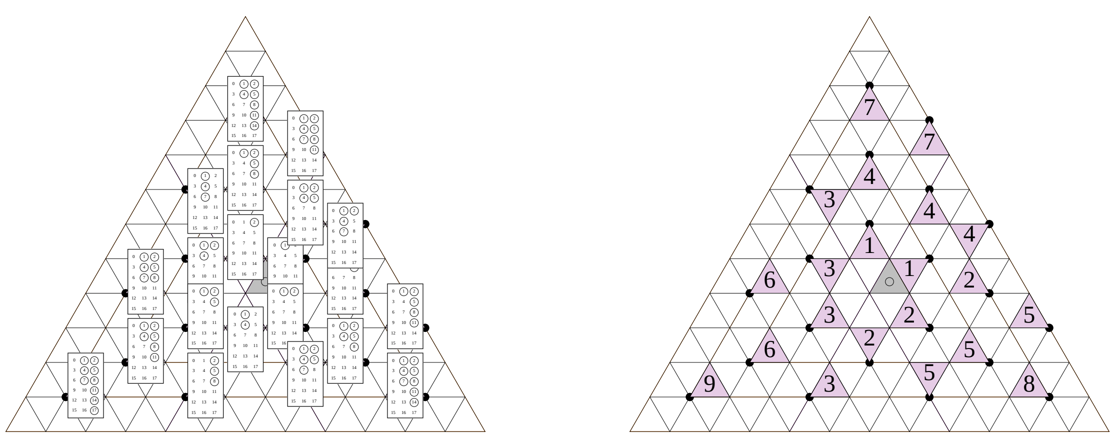

FIGURE 15. The abaci corresponding to inverse alcoves have Ish number of beads.

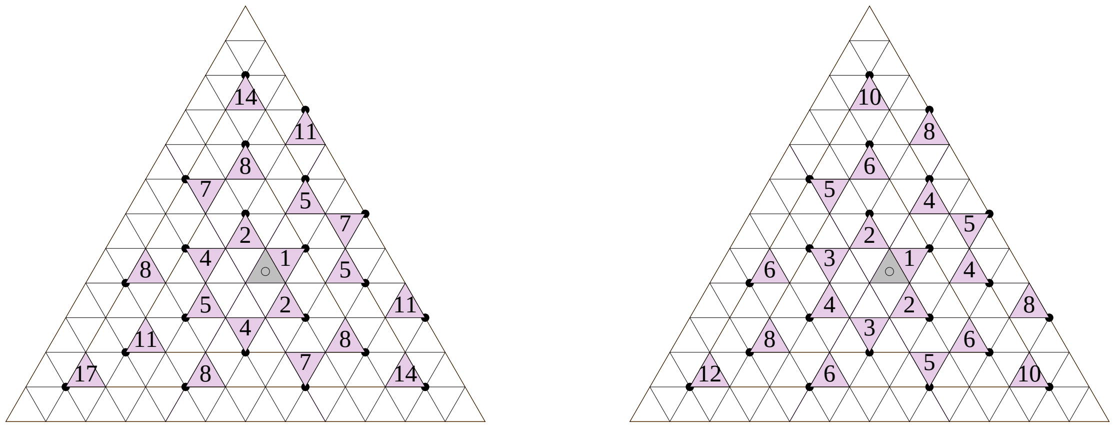

FIGURE 16. The lowest bead number for an abacus (left) corresponds to a modified Shi number. Shi numbers are on the right.

the vectors that are perpendicular, $h_1 = (\sqrt{3}/6, 1/2)$, and $h_2 = (\sqrt{3}/3, 0)$:

$$\text{Ish}(\mathbf{x}) = \begin{cases} 6\langle h_1, \mathbf{x}\rangle - 2 & \text{when } \langle \alpha_2, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle > 0 \\ -6\langle h_2, \mathbf{x}\rangle & \text{when } \langle \alpha_1, \mathbf{x}\rangle \leq 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle \leq 0 \\ -6\langle h_1 - h_2, \mathbf{x}\rangle - 1 & \text{when } \langle \alpha_1, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_2, \mathbf{x}\rangle \leq 0 \end{cases}$$

$$\text{Shi}(\mathbf{x}) = \begin{cases} 4\langle \alpha_1 + \alpha_2, \mathbf{x}\rangle - 3 & \text{when } \langle \alpha_1, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_2, \mathbf{x}\rangle > 0 \\ 4\langle \alpha_1, \mathbf{x}\rangle - 2 & \text{when } \langle \alpha_1, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_2, \mathbf{x}\rangle \leq 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle > 0 \\ 4\langle \alpha_2, \mathbf{x}\rangle - 2 & \text{when } \langle \alpha_2, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_1, \mathbf{x}\rangle \leq 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle > 0 \\ -4\langle \alpha_2, \mathbf{x}\rangle - 1 & \text{when } \langle \alpha_1, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle \leq 0 \\ -4\langle \alpha_1, \mathbf{x}\rangle - 1 & \text{when } \langle \alpha_2, \mathbf{x}\rangle > 0 \text{ and } \langle \alpha_1 + \alpha_2, \mathbf{x}\rangle \leq 0 \\ -4\langle \alpha_1 + \alpha_2, \mathbf{x}\rangle & \text{when } \langle \alpha_1, \mathbf{x}\rangle \leq 0 \text{ and } \langle \alpha_2, \mathbf{x}\rangle \leq 0 \end{cases}$$

I did not prove them; I inferred them from the data. The modified Shi number is calculated by counting Shi possible bead positions in the abacus, or as a formula, $\widetilde{\text{Shi}} = \lfloor \frac{\text{Shi}-1}{3} \rfloor + (\text{Shi}-(1 \bmod 2)) + 1$. Now that I have these formulas written down, they do not seem pleasant at all to work with and calculate $\widetilde{\text{Shi}}-\text{Ish}$. Unfortunately, after all this, I think this simply returns to the method of calculating the number of boxes from Section 1—why use abaci to calculate the coordinates of alcoves to calculate the number of separating hyperplanes when we could calculate the number of separating hyperplanes directly? I do like the connection of the abacus diagram to Shi and Ish.

One note: We can see in Figure 14b what Monica mentioned in Iceland—that by reflecting an alcove that is "distance k" from the hyperplane $H_{\alpha,0}$ adds (or subtracts) exactly k boxes to the corresponding core.

Last, a question for Drew. You wrote back in August:

> It is not too difficult . . . to show that $D^p(n)$ contains p^{n-1} alcoves, and contains $\frac{1}{n+p}\binom{n+p}{n,p}$ elements of the root lattice, which are in bijection with (n,p)-cores.

Can you share the bijection of (n,p)-cores with the elements of the root lattice in $D^p(n)$?

I thank you for all your help and your patience with me as I learn the material; I hope it has not been too much of a bother.

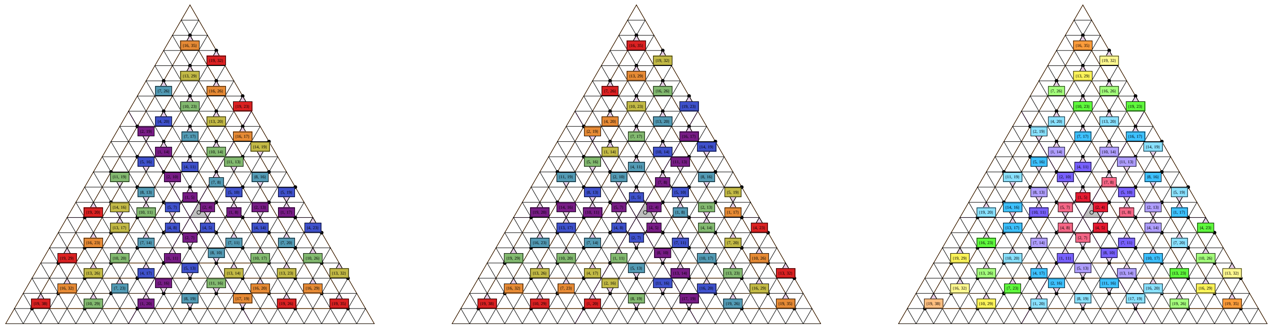

FIGURE 17. Colors correspond to equivalent alcove coordinates. (a) Colors based on minimum bead. (b) Colors based on difference between beads. (c) Colors based on maximum bead.

16. CH, 18 October 2011

Ignore my last question—you answered it in September. I need to sit down and actually write out those relations as coming from the abaci.

16.1. Thoughts on numerology.

Drew anticipates that the size of the average simultaneous core is $(n+p+1)(n-1)(p-1)/24$. This formula is highly related to other numbers that occur in simultaneous cores.

- $(n^2-1)(p^2-1)/24$ is the size of the largest simultaneous core. Notice that $(n^2-1)(p^2-1)/24 = (np+n+p+1)(n-1)(p-1)/24$, which means that the average simultaneous core is $(n+p+1)/(np+n+p+1) = \big(1/(n+1)+1/(p+1)-1/(n+1)(p+1)\big)$ times the size of the largest simultaneous core.
- $\binom{n+p}{p}/(n+p)$ is the number of simultaneous cores, so this means that the total number of boxes in all simultaneous cores needs to be $\frac{(n+p+1)(n-1)(p-1)}{24(n+p)}\binom{n+p}{p}$, which has multiple equivalent forms, but none as symmetric. However, it can be massaged into $\frac{(n^2-1)(p^2-1)}{24}\frac{1}{(n+p)(n+p+2)}\binom{n+p+2}{p+1}$, which is a multiple of the size of the largest simultaneous core. (Careful that $(n+p)(n+p+2)$ might not divide evenly into $\binom{n+p+2}{p+1}$.)

16.2. Thoughts on the previous section.

There is little need to discuss $\widetilde{\mathrm{Shi}}(\mathbf{x})$ instead of simply $\mathrm{Shi}(\mathbf{x})$ because we end up taking $\lceil \widetilde{\mathrm{Shi}}(\mathbf{x})/3 \rceil = \lceil \mathrm{Shi}(\mathbf{x})/2 \rceil$. I created pictures to visualize the occurrences of values of k as coordinates by looking at the inverse picture. (See Figure 17.) It was surprising for me to see the symmetry involved, especially how the abaci corresponding to having a very low bead are centered around three rays propagating from the center, instead of three lines through the origin. Also remember that the max bead corresponds to Shi, so Figure 17(c) is the same as Figure 16(b).

16.3. Thoughts on finding a function for red data.

I thought about trying to get a better formula for the red data. It still looks ugly but less intimidating. For example, in Case 1 with alcove coordinates $\begin{bmatrix} i+j+k+1 & i+j & i \\ j+k & j & \\ k & & \end{bmatrix}$, we can ask how many choices for (i,j,k) satisfy $i+j+k=x$, and this is simply a multichoose, so it is $\binom{x+2}{2}$, with generating function $1/(1-x)^3$ (truncated). Because we have the coordinate $i+j+k+1$ instead of $i+j+k$, we end up shifting these by x.

Also, any set of coordinates for $m-1$ are also a set of coordinates for m, so we might as well focus on what is added for m.

17. CH, 21 October 2011

17.1. **Beta numbers.**

Here's an approach that seems a bit more promising. I was thinking about trying to determine the size of of the core partition directly from the abacus diagrams. While I remembered the idea of counting bead-gap pairs, it took some rereading of other papers to reprocess the idea of beta numbers. For myself, let me recall that when the abacus is normalized (first gap at position 0), then if we write down the positive beads in decreasing order as $(\beta_1, \beta_2, \dots, \beta_b)$, these are called the "beta numbers" (what a horrible name—perhaps I will rename them the bead numbers). These bead numbers correspond to the hook lengths of the boxes in the first column of the corresponding partition λ, and as such, we can write down $\lambda = (\beta_1 - (b-1), \beta_2 - (b-2), \dots, \beta_{b-1} - 1, \beta_b - 0)$.

This means that the total number of boxes in $\lambda = \sum_{1\le i\le b} \beta_i - \binom{b}{2}$. Since b is the number of beads in the abacus, we now know that this equals $\mathrm{Ish}(\lambda)$. In simple cases, the calculation of $\displaystyle\sum_{\text{all abaci}} \Bigg(\sum_{1\le i\le b} \beta_i\Bigg)$ should not be impossible to figure out by appealing to Anderson's lattices. For example, we can find that for $(3, 3m+1)$-simultaneous cores, we know that we are allowed to have any normalized abacus of $(0, 3i+1, 3j+2)$ satisfying $0 \le i \le m$ and $0 \le j \le m+i$, so the sum of all bead numbers is

$$\sum_{i=0}^{m}\sum_{j=0}^{m+i}\left[\sum_{I=0}^{i}(3I+1) + \sum_{J=0}^{j}(3J+2)\right] = \frac{m}{12}(1+m)(1+3m)(10+11m).$$

Similarly, for $(4, 4m+1)$-simultaneous cores, the sum of all bead numbers is

$$\sum_{i=0}^{m}\sum_{j=0}^{m+i}\sum_{k=0}^{m+j}\left[\sum_{I=0}^{i}(4I+1) + \sum_{J=0}^{j}(4J+2) + \sum_{K=0}^{k}(4K+3)\right] = \frac{1}{6}(1+m)\big(14m+91m^2+164m^3+88m^4\big).$$

If we know the distribution of Ish in $D^p(n)$, then we could find $\sum_{a\in D^p(n)} \binom{\mathrm{Ish}(a)}{2}$, and know the total number of boxes of all cores in $D^p(n)$. For $(3, 3m+1)$, it *appears* that the distribution of Ish is $\big(0^1, 1^2, 2^3, \dots, m^{m+1}, (m+1)^m, (m+2)^m, (m+3)^{m-1}, \dots, (3m-1)^1, (3m)\big)$, so that the quantity we need to remove from the sum of all bead numbers is

$$\sum_{i=0}^{m}(i+1)\binom{i}{2} + \sum_{i=1}^{m}(m-i+1)\left[\binom{m+2i-1}{2} + \binom{m+2i}{2}\right] = \frac{m}{24}(1+m)(-10+19m+39m^2),$$

giving us that the total number of boxes in all $(3, 3m+1)$ simultaneous cores is exactly $m(1+m)(2+3m)(5+3m)/8$, which is correct. It does not appear that the distribution of Ish in $(4, 4m+1)$ cores is as nice.

Question 17.1. *Is there a generating function for* $\displaystyle\sum_{a\in D^p(n)} \mathrm{Ish}(a)$*? Or is something known about its distribution, or better, the distribution of* $\binom{\mathrm{Ish}(a)}{2}$*?*

Perhaps it is possible to work these two sums together in such a way that the answer falls out nicely. Perhaps some way to break down the Ish statistic further and fit it into the sums...

17.2. **Other abacus observations.**

Earlier in the day, I looked at balanced abaci instead of normalized abaci, and investigated the pictures of the abaci placed on dominant alcoves and inverted alcoves. In the inverted alcove picture, I saw the known bijection between levels of the lowest beads and the root coordinates, that at coordinate $(x\alpha_1 + y\alpha_2)$, the abacus is $(-3y+3, 3x+1, 3(-x+y)+2)$, although I do not understand why the first coordinate has a negative sign, nor do I understand the appearance of $-x+y$. This made me realize that given the coordinates of the root, we can easily find the lowest beads in each runner. I suppose I am just a few steps away from understanding the appearance of $D^p(n)$.

When I looked at the dominant alcoves, I saw that there was a different method of understanding the ways in which the beads change when you walk between alcoves. When you walk to the right, it adds one to the highest bead. When you walk to the northeast, it subtracts one from the lowest bead. Some steps are both to the right and to the northeast, and those are the ones which interchange those two runners, accomplishing adding one and subtracting one at the same time.

17.3. **Investigating Subdiagrams.**

I also wanted to investigate Fayers's result (actually Vandehey's) that every (n, p) simultaneous core is a subdiagram of the maximal (n, p) simultaneous core κ. What I investigated is that there appears to be a bijection between the boxes in the Durfee square of κ and simultaneous core partitions that satisfy certain hyperplane inequalities. By choosing any box in the Durfee square, take all boxes in κ to the right and below, and this is another simultaneous core partition. But there is no bijection between the cores that don't satisfy those hyperplane inequalities and other subdiagrams—there are some diagrams that appear multiple times as south-east subdiagrams of κ and there are some diagrams that **do not** appear as south-east subdiagrams of κ.

I realized that these south-east subdiagrams that have a box in the Durfee square of κ correspond to removing gaps from the bottom of an abacus or removing beads from the top of an abacus. What was happening was that the "nice" subdiagrams correspond to abacus diagrams (a_0, a_1, a_2) satisfying $a_0 \leq a_1 \leq a_2$, and the "not nice" subdiagrams correspond to when $a_1 > a_2$. In both of these cases, it is possible to count bead-gap pairs in an easy way. This made me think that perhaps there would be an easy way to count bead-gap pairs when there are more runners, conditioning on the exact permutation of the lengths of the runners.

18. DA, MV, 21–22 October 2011

18.1. **Monica writes:**

> This might not be quite what you meant, but ish is related to the length (or width) of cores. Let's say width, so λ_1. Chris Berg and I studied the generating function of cores w.r.t. the stat of λ_1 and it's in [BV08]. It's a pretty simple function. How that fits in with just summing over $D^p(n)$, I'm not sure since I have to read the backlog for the notation.

18.2. **Drew writes:**

Conjecturally, the distribution of ish on $D^p(n)$ is equal to the distribution of shi on $D^p(n)$. (For shi to be defined we currently need to assume $p = \pm 1 \mod n$.) And this distribution of shi is given by the rank numbers of the poset of dominant (bounded or arbitrary) m-Shi regions, which in a sense we know, but I don't think there's a closed formula. In the case $m = 1$ we are looking at the poset of Dyck paths under inclusion of Ferrers shapes. Do the rank numbers of this poset have a closed formula? (I don't know.)

19. CH, 26 October 2011

19.1. **Notes on ish.**

I learned that ish is **not** equal to the number of beads in the abacus. Working from the definition in [Arm09], I see that in fact, ish equals the number of *gaps* before the lowest bead in the abacus. Since there is not a nice relationship in general (when $p \neq mn+1$) between the number of gaps before the lowest bead in the abacus and the the number of beads after the highest gap in the abacus, it is not the case that ish equals the number of beads in the abacus.

The picture that had inspired this misunderstanding (Figure 15) should instead have been reflected across a line through the origin. I had stumbled upon an injection in the set of simultaneous $(3, 3m+1)$-core partitions between abaci with k beads and abaci with k gaps before the last bead. This does not happen when $n > 3$, but perhaps there is something of interest to study here.

In any case, this implies that when calculating the number of beads in an abacus (in order to reconcile a sum of bead numbers), the ish statistic is a red herring.

I had tried to approach finding a formula for the ish statistic using Monica's reference of her work with Chris Berg. It does appear that something vaguely similar is possible. Instead of finding a bijection between $\left\{ \, l\text{-cores with first part } \lambda_1 \, \right\}$ and $\left\{ \, (l-1)\text{-cores with first part } \leq \lambda_1 \, \right\}$, it makes sense to look for a bijection between and $\left\{ \, (n,p)\text{-cores with first part } \lambda_1 \, \right\}$ and $\left\{ \, (n,p)\text{-cores with first part } \leq \lambda_1 \text{ AND no bead in runner } (-p) \mod n \, \right\}$. The reason for these unpleasantries has to do with the necessity for the lattice path to stay under a line with slope n/p.

19.2. **Positive data.**

I doubt that you had any doubt about the veracity of your conjecture, but *Mathematica* crunched some sums and found that your conjecture holds for **all** $(3, 3m+1)$, $(4, 4m+1)$, through $(10, 10m+1)$ simultaneous core partitions. For example, to prove $(6, 6m+1)$ simultaneous cores, we calculate

$$\sum_{i_1=0}^{m} \sum_{i_2=0}^{m+i_1} \cdots \sum_{i_5=0}^{m+i_4} \left[\sum_{j_1=0}^{i_1} (6j_1+1) + \cdots + \sum_{j_5=0}^{i_5} (6j_1+5) - \binom{i_1+\cdots+i_5}{2} \right]$$

(# beads $= i_1 + i_2 + i_3 + i_4 + i_5$), which sums to

$$\frac{m}{24 \cdot 4!}(2+6m)(3+6m)(4+6m)(5+6m)(6+6m)(8+6m);$$

dividing by $\binom{6m+7}{6}/(6m+7) = (6m+6)_5/6!$ gives $(6m+1+6+1)(6-1)(6m)/24$.

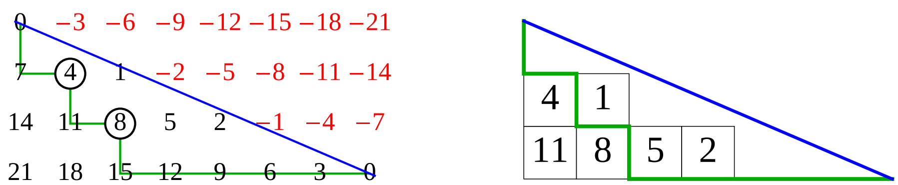

FIGURE 18. A lattice path from $(0,3)$ to $(7,0)$ staying below $3-\frac{3}{7}x$. The left figure is the original conception given in Figure 9; the right figure is discussed in Section 19.3.

After some necessary retooling, I was able to similarly show that your conjecture holds for **all** $(3,3m-1)$, $(4,4m-1)$, through $(10,10m-1)$ simultaneous core partitions. For example, for $(7,7m-1)$ simultaneous cores, we calculate

$$\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m+i_1}\cdots\sum_{i_5=0}^{m+i_4}\left[\sum_{j_1=0}^{i_1}(7j_1+6)+\cdots+\sum_{j_6=0}^{i_6}(7j_1+1)-\binom{i_1+\cdots+i_6}{2}\right],$$

which simplifies to

$$\frac{m}{24\cdot 5!}(-2+7m)(1+7m)(2+7m)(3+7m)(4+7m)(5+7m)(7+7m).$$

Dividing by $\binom{7m+6}{7}/(7m+6)=(7m+5)_6/7!$ gives $(7m-1+7+1)(7-1)(7m-2)/24$.

I can also calculate the number of boxes for all simultaneous (n,p)-cores for specific values of n and p. But looking at this data says that somewhere there is some major canceling going on! When I look at these answers for the number of boxes and I see a product of factors that are so similar, it implies to me that we should be able to count the number of boxes very simply using independent factors. We can see that there should be n factors, each of which has the form $p\pm x$ for some x!!!!!! I'm sure it will just pop out some day and we'll say 'Duh!'.

19.3. **Lattice paths.**

One way I went about this is to revert to Anderson's lattice paths. I realized that I should probably be drawing these lattice paths as in Figure 18(b). In this new drawing, the lattice path is a separation between possible positions with beads and possible positions without beads. We conclude that the number of beads in the abacus is the area between the lattice path and the diagonal line. So if there is some way to understand the area statistic for the set of all lattice paths then this is the quantity that we need to remove from the sum of all beta numbers.

In addition, the beta numbers are exactly the numbers inside the boxes that are between the lattice path and the diagonal line. Perhaps an understanding of the area statistic will lead to an understanding of the distribution of these beta numbers.

Question 19.1. *Is anything known about the* area *statistic for lattice paths from* $(0,0)$ *to* (p,n) *that stay above the line* $ny=px$*?*

A natural place to start to understand the area statistic is in the work of Nick Loehr, but he only addresses area for lattice paths from $(0,0)$ to (mn,n). (And in that vein, he only addresses it using a joint distribution.) I have been unable to locate information about the single-variable distribution of the area statistic for these lattice paths, which must be easier and is hopefully generalizable.

Another avenue of approach is from the point of view of "ballot sequences", which encompasses all sorts of questions related to lattice paths satisfying a restriction. I imagine that once we learn about the area statistic in this case, we might be able to relate it back to the literature in ballot sequences.

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