Maximal Planar Graphs

A graph with “too many” edges isn’t planar; how many is too many?

**Goal:** Find a numerical characterization of “too many”

**Definition:** A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

**Examples:** Octahedron \( K_4 \) \( K_5 \setminus e \)

What do we notice about these graphs?
Numerical Conditions on Planar Graphs

Every face of a maximal planar graph is a triangle!

**Theorem 8.1.2:** If \( G \) is maximal planar, then \( q = 3p - 6 \).

**Proof:** In any plane drawing of \( G \), let \( p = \# \) of vertices, \( q = \# \) of edges, and \( r = \# \) of regions. We will count the number of face-edge incidences in two ways.

From a face-centric POV, the number of face-edge incidences is

From an edge-centric POV, the number of face-edge incidences is

Substitute into Euler’s formula:

Every planar graph is a subgraph of a maximal planar graph.

Every maximal planar graph has exactly \( q = 3p - 6 \) edges.

**Cor 8.1.3:** Every planar graph with \( p \) vertices has at most \( 3p - 6 \) edges!
Numerical Conditions on Planar Graphs

**Theorem 8.1.4:** The graph $K_5$ is not planar.

**Proof:**

**Theorem 8.1.5:** If $G$ is planar with girth $\geq 4$, then $q \leq 2p - 4$.

**Proof:** Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

**Theorem 8.1.5:** If $G$ is planar and bipartite, then $q \leq 2p - 4$.

**Theorem 8.1.6:** $K_{3,3}$ is not planar.

**Theorem 8.1.7:** Every planar graph has a vertex with degree $\leq 5$.

**Proof:**
Dual Graphs

**Definition:** Given a plane drawing of a planar graph $G$, the **dual graph** $D(G)$ of $G$ is a graph with vertices corresponding to the regions of $G$. Two vertices are connected by an edge each time the two regions share an edge as a border.

- The dual graph of a simple graph may not be simple.
- Two regions may be adjacent multiple times.
- $G$ and $D(G)$ have the same number of edges.

**Definition:** A graph $G$ is **self-dual** if $G$ is isomorphic to $D(G)$. 
Maps

*Definition:* A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a *normal map*.

*Definition:* In a map, the regions are called *countries*. Countries may share several edges.

*Definition:* A *proper coloring* of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

*Question:* How many colors are necessary to properly color a map?
Proper Map Colorings

Lemma 8.2.2: If $M$ is a map that is a union of simple closed curves, the regions can be colored by two colors.

Proof: Color the regions $R$ of $M$ as follows:

\[
\begin{align*}
\text{black} & \quad \text{if } R \text{ is enclosed in an odd number of curves} \\
\text{white} & \quad \text{if } R \text{ is enclosed in an even number of curves}
\end{align*}
\]

This is a proper coloring of $M$. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.
Lemma 8.2.6: (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

Proof: Very hard.

★ This is the wrong object ★

Theorem: If $G$ is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

- Every face of $G$ is a triangle $\rightsquigarrow$
- $G$ is connected $\rightsquigarrow$
- $G$ is planar $\rightsquigarrow$
Assuming Lemma 8.2.6,

\[ G \text{ is maximal planar} \implies D(G) \text{ is a normal map} \]
\[ \implies \text{countries of } D(G) \text{ 4-colorable} \]
\[ \implies \text{vertices of } G \text{ 4-colorable} \]
\[ \implies \chi(G) \leq 4 \]

This proves

**Theorem 8.2.8:** If \( G \) is maximal planar, then \( \chi(G) \leq 4 \).

Since every planar graph is a subgraph of a maximal planar graph, Lemma C implies:

**Theorem 8.2.9:** If \( G \) is a planar graph, then \( \chi(G) \leq 4 \).