Math 308U

Quiz 3 Solutions

December 1, 2004

Question 1: (12 points) Calculate the best *linear* approximation of the following data using the method of least squares.

t	-1	1	2	4
y	4	1	5	-2

Answer: We want to solve the normal equations $A^TA\mathbf{x} = A^T\mathbf{y}$, where

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 4 \end{bmatrix},$$

and $\mathbf{y} = [4, 1, 5, -2]^T$. Calculating A^TA gives

$$A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 22 \end{bmatrix},$$

and calculating $A^T\mathbf{y}$ gives $A^T\mathbf{y} = [8, -1]$. Solving for $\mathbf{x}$, we reduce the augmented matrix

$$\begin{bmatrix} 4 & 6 & 8 \\ 6 & 22 & -1 \end{bmatrix},$$

giving

$$\begin{bmatrix} 1 & 0 & 3.5 \\ 0 & 1 & -1 \end{bmatrix},$$

which implies the best linear approximation to the data is the line $y = (-1)x + 3.5$.

Question 2a: (5 points) Calculate the eigenvalues for the following matrix.

$$A = \begin{bmatrix} 0 & 3 & 6 \\ 0 & 4 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer: We need to take the determinant of the matrix

$$A - \lambda I = \begin{bmatrix} -\lambda & 3 & 6 \\ 0 & 4-\lambda & -2 \\ 0 & 1 & 1-\lambda \end{bmatrix}.$$

Doing a cofactor expansion about the first column gives $(-\lambda)(-1)^{1+1}\det M_{11}$. The determinant of the (1,1)-minor matrix of A is $(4-\lambda)(1-\lambda) - (-2) = \lambda^2 - 5\lambda + 6$, which factors to $(\lambda - 3)(\lambda - 2)$. The eigenvalues are the values λ such that $\det A = 0$, so this is exactly when $\lambda = \{0, 2, 3\}$.

Question 2b: (6 points) Find an eigenvector for A corresponding to the largest eigenvalue you found in Question 2a.

Answer: The largest eigenvalue is when $\lambda = 3$. Plugging in $\lambda = 3$ to the matrix in the Answer to Question 2a, we get

$$A - 3I = \begin{bmatrix} -3 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}.$$

Solving $(A - 3I)\mathbf{x} = \theta$ gives $\mathbf{x} = x_3[4, 2, -1]^T$, so an eigenvector of A corresponding to $\lambda = 3$ is $\mathbf{x} = [4, 2, -1]^T$.

Question 3: Quick Answer Questions (QuAQ's)
(a) (3 points) Give an example of a 3×3 triangular matrix that is not invertible.

Answer: Give any triangular matrix that has at least one zero on its diagonal.

(b) (3 points) If A and B are 4×4 matrices such that $\det A = 3$ and $\det B = 8$, find the value of $\det(A^{-1}B^2)$.

Answer: $\det(A^{-1}B^2) = \det(A^{-1})\det(B^2) = (\det B)^2/\det(A) = 64/3$.

(c) (3 points) True or False: For every linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $x \in \mathbb{R}^m$.

Answer: False. The matrix A is $n \times m$. You could check that an $m \times n$ matrix times an $m \times 1$ vector is not well defined.

(d) (3 points) True or False: If $\mathbf{x}$ is a solution to the system of equations $A\mathbf{x} = \mathbf{b}$, then the residual vector corresponding to $\mathbf{x}$ is equal to the zero vector θ.

Answer: True. The residual vector is $\mathbf{b} - A\mathbf{x}$. If $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b} - A\mathbf{x} = \theta$.