Math 308U

Midterm Answer Key

November 10, 2004

Question 1: (8 points) For what values of λ is the following set of vectors linearly independent?

$$\left\{ \begin{bmatrix} \lambda - 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda - 2 \end{bmatrix} \right\}$$

Answer: Two solutions. First, the columns of a matrix are linearly dependent if the determinant of the matrix is zero. This means that $(\lambda - 2)^2 - 1 = 0$. So the columns are linearly independent if λ is not equal to 3 or 1.

Second, you can row reduce the matrix

$$\begin{bmatrix} 1 & \lambda - 2 \\ \lambda - 2 & 1 \end{bmatrix}$$

in which case you get

$$\begin{bmatrix} 1 & \lambda - 2 \\ 0 & 1 - (\lambda - 2)^2 \end{bmatrix}$$

The columns are linearly independent if $1-(\lambda-2)^2$ is not equal to zero, producing the same answer, λ not equal to 1,3.

Question 2: (8 points) Define the set S of vectors to be

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 8 \\ 0 \end{bmatrix} \right\}.$$

Let W be the subspace of $\mathbb{R}^4$ that these vectors span. Prove that S is an orthogonal basis for the subspace W.

Answer: It is easy to check that $\mathbf{v}_i^T \mathbf{v}_j = 0$ for each pair i, j. Therefore the set if orthogonal. Next you must show that this set is a basis. To do this, you must show that the vectors are linearly independent. Since the vectors are orthogonal, they are linearly independent, so S is an orthogonal basis of W.

Question 3: (4 points) Let S and W be defined as in the previous question. Find an *orthonormal* basis for W.

Answer: Divide each v by its length. $\mathbf{w}_1 = \frac{1}{\sqrt{6}}\mathbf{v}_1$, $\mathbf{w}_2 = \frac{1}{\sqrt{21}}\mathbf{v}_2$, and $\mathbf{w}_3 = \frac{1}{\sqrt{72}}\mathbf{v}_3$.

Question 4: (6 points) Is the set $W = \{\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \cdot x_2 \geq 0\}$ a subspace of $\mathbb{R}^2$? If so, prove that all three subspace properties hold. If not, give an example of a violation of one of the subspace properties. (Hint: *Draw a picture of* W.)

Answer: The set W is not a subspace of $\mathbb{R}^2$. Condition 2 is violated. If $\mathbf{u} = [1, 2]^T$ and $\mathbf{v} = [-2, -1]^T$ are both in W, then $\mathbf{u} + \mathbf{v} = [-1, 1]^T$ is not in W.

Question 5: (6 points) A matrix A is called *skew-symmetric* if $A^T = -A$. Prove that if the $n \times n$ matrix A is skew symmetric, then every entry a_{ii} along the diagonal of A is zero.

Answer: Consider entry (i, i) of A^T. It is equal to $a_{i,i}$. Consider entry (i, i) of $-A$. It is equal to $-a_{i,i}$. Since these two matrices are equal, then $a_{i,i} = -a_{i,i}$, which implies that $a_{i,i} = 0$ for all i.

Question 6: (8 points) Find the rank, the nullity, and the dimension of the row space of the following matrix

$$A = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 1.5 & 2 \end{bmatrix}$$

Answer: Row reduction gives

$$\begin{bmatrix} 1 & 1.5 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore the rank of the matrix is 1. The dimension of the row space of the matrix is therefore also 1. By the rank nullity law, the nullity of the matrix is $n - r = 3 - 1 = 2$.

Question 7: Quick Answer Questions (QuAQ's)

(a) (4 points) If $B = P^{-1}AP$, write B^{100} in terms of A and P.

Answer: $P^{-1}A^{100}P$.

(b) (2 points) True or False: The following matrix is invertible.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 2 & 4 & 8 \\ -3 & 6 & 2 \end{bmatrix}$$

Answer: False, the matrix is not $n \times n$.

(c) (2 points) True or False: If $A^2 = -I$, then A is invertible.

Answer: True. Note that $-A$ is the inverse for A. Since A has an inverse, it is invertible.

(d) (2 points) True or False: If the rank of an $n \times n$ matrix is n, then the matrix is singular.

Answer: False. A full rank matrix is **non**-singular.