A proof of Problem 2.4.2

Question: Show that if 100 integers are chosen from $1, 2, \ldots, 200$, and one of the integers chosen is less than 16, then there are two chosen numbers such that one of them is divisible by the other.

Answer: Assume that there is a selection of 100 integers such that no two divide one another. As in Application 5, write the 100 integers b_i in the form $2^k a_i$, for an odd number a_i (Factor out all the powers of two). Since no two integers b_i and b_j divide one another, the 100 a_i all are distinct, so must be the numbers $1, 3, 5, \ldots, 199$. We consider the possible values for the one integer b^* that is less than 16, and show that in each of those cases, there must be two integers that divide each other.

Case 1: $b^* = 1, 3, 5, 7, 9, 11, 13$, or 15. Notice that $3b^*$ is an odd number less than 199. Since there is an integer b_i with odd part $a_i = 3b^*$, then $b^* | b_i$.

Case 2: $b^* = 2, 6, 10$, or 14. In these cases, $b^* = 2a^*$ for $a^* = 1, 3, 5$, or 7. Consider $a' = 3a^*$ and $a'' = 9a^*$, both odd integers less than 199. The set of integers $\{b^*, b', b''\}$ (with odd parts $\{a^*, 3a^*, 9a^*\}$) contains a set of two integers, one of which divides the other.

Case 3: $b^* = 4$ or 12. In these cases, $b^* = 2^2 a^*$ for $a^* = 1$ or 3. Consider $a' = 3a^*$, $a'' = 9a^*$, and $a''' = 27a^*$, all odd integers less than 199. The set of integers $\{b^*, b', b'', b'''\}$ (with odd parts $\{a^*, a', a'', a'''\}$) contains a set of two integers, one of which divides the other.

(Work out on your own why the statements at the end of Case 2 and Case 3 are valid. Come and see me if you can not figure it out!)