Connectivity Definitions and Theorems from Weeks 3 and 4

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Connectivity Definitions:

Definition: A graph G is connected if for all $v, w \in V(G)$, there exists a path from v to w.
Definition: G is disconnected if G is not connected.
Definition: A (connected) component H is a MAXIMAL subgraph H of G that is connected.
Definition: G is k-connected if $|V(G)| > k$ and removing fewer than k vertices does not disconnect the graph.

(We say that every graph is 0-connected.)

Definition: The connectivity of G (denoted $\kappa(G)$ = "kappa") is the maximum k such that G is k-connected.

(Conventions: The connectivity of a single vertex is zero and $\kappa(K_n) = n - 1$.)

Definition: A cut vertex is a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected.
Definition: A SEPARATING SET or vertex cut is a set of vertices $X \subset V(G)$ such that $G \setminus X$ is disconnected.
Note: $\kappa(G) = 0 \iff G$ is disconnected or G is a single vertex
Note: $\kappa(G) \geq 2 \iff G$ has no cut vertex.

Edge Connectivity Definitions:

Definition: G is k-edge-connected if removing fewer than k edges does not disconnect the graph.

(We say that every graph is 0-edge-connected.)

Definition: The edge connectivity of G (denoted $\lambda(G)$ = "lambda" or $\kappa'(G)$) is the maximum k such that G is k-edge connected.
Definition: A bridge is an edge $e \in E(G)$ such that $G \setminus e$ is disconnected.
Definition: A DISCONNECTING SET is a set of edges $D \subset E(G)$ such that $G \setminus D$ is disconnected.
Note: $\lambda(G) = 0 \iff G$ is disconnected.
Note: $\lambda(G) \geq 2 \iff G$ has no bridge.

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Note: If you delete a cut vertex from a graph, the number of connected components increases.
Note: If you delete a bridge from a graph, the number of connected components increases by exactly one.

Theorems:

Theorem: (Book 2.4.1) Let G be connected. Then G is a tree $\iff$ Every edge of G is a bridge.
Theorem: (Book 3.2.1) A regular graph of even degree has no bridge.
Theorem: For all graphs G, $\lambda(G) \leq \delta(G)$.

Edge Cuts: [not to be confused with cutset or cut vertex]

Definition: Let $X \subset V(G)$. Then X^c is the complement of X.
That is, $V(G) = X \cup X^c$ and $X \cap X^c = \emptyset$.
Definition: For any $X \subset V(G)$ such that $X, X^c \neq \emptyset$, an edge cut (denoted $[X, X^c]$) is the set of edges D between X and X^c.
Note: An edge cut is a disconnecting set, but not vice versa.
(This implies $\lambda(G) \leq \big|[X, X^c]\big|$ for all $X \subset V(G)$.
Note: A minimal disconnecting set is an edge cut, but not vice versa.
Theorem: For all graphs G, $\kappa(G) \leq \lambda(G)$.

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Blocks:

Definition: A block of a graph G is a maximally connected subgraph of G with no cut vertex.
Note: The following things are true about blocks.

1. G itself may be a block.
2. Except for blocks that are edges, blocks are always 2-connected.
3. Any two blocks share at most one vertex.
4. A vertex shared between blocks is a cut vertex of G.
5. The blocks of G partition $E(G)$.

Definition: The block graph of G is a bipartite graph H with vertices v_i representing cut vertices of G, and vertices b_j representing blocks of G, where $v_i b_j$ is an edge of H if vertex v_i is a vertex in block b_j.
Note: A block graph is always a forest. (Proof in hwk.)

More Graph Statistics:

Definition: An independent set of a graph G is a subset $X \subset V(G)$ such that no edge of G connects any two vertices of X. In other words, the induced subgraph of G on X contains no edges.
Definition: The independence number of a graph G is the size of the maximum independence set of G. It is denoted $\alpha(G)$.
Definition: A vertex cover of a graph G is a subset $X \subset V(G)$ such that X contains (at least) one endpoint of every edge in G.
Definition: The size of the smallest vertex cover is denoted $\beta(G)$.
Theorem: In any graph G, $X \subset V(G)$ is an independent set $\iff$ X^c is a vertex cover.
Theorem: For all graphs G, $\alpha(G) + \beta(G) = |V(G)|$.
Note: Finding an independent set X and a vertex cover Y such that $|X| + |Y| = |V(G)|$ implies that $\alpha(G) = |X|$ and $\beta(G) = |Y|$.
Definition: The clique number $\omega(G)$ of a graph G is the largest number k such that K_k is a subgraph of G.

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Characterization of 2-connectedness:

Theorem: (Whitney, 1932 aka MINI-Menger) Let G be a graph with ≥ 3 vertices. Then G is 2-connected $\iff$ for all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.
Theorem: (Menger) G is k-connected $\iff$ for all $v, w \in V(G)$, there exist k internally disjoint v, w-paths in G.
Definition: Let H be any subgraph of G. Then an H-path (or an ear) is a path in G that starts and ends in H.
Definition: An ear decomposition is a construction of G starting with some cycle C, and at each step successively adding to the existing graph H some H-path.
Theorem: G is 2-connected $\iff$ G has an ear decomposition.
Theorem: Let G be a graph with ≥ 3 vertices. The following are equivalent:

1. G is 2-connected.
2. G is connected and has no cut vertex.
3. G is a block.
4. For all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.
5. For all $v, w \in V(G)$, there exists a cycle in G through v and w.
6. $\delta(G) > 0$ and for all $e, f \in E(G)$, there exists a cycle in G through e and f.
7. G has an ear decomposition.

List of graph statistics so far

$\alpha(G)$

$\beta(G)$

$\delta(G)$

$\Delta(G)$

$\kappa(G)$

$\lambda(G)$

$\omega(G)$

$diam(G)$

$g(G)$