

where  $d_{i,j}$  ranges over all of the positive common divisors of  $i$  and  $j$ . But then by (1.23) we have

$$\sum \phi(d_{i,j}) = (i, j),$$

whence

$$A\Phi A^T = B.$$

The  $(0,1)$ -matrix  $A$  of order  $n$  is triangular with 1's in the  $n$  main diagonal positions. Since the determinant function is multiplicative it follows that

$$\det(B) = \det(\Phi).$$

□

### Exercises

1. Let  $A$  be a  $(0,1)$ -matrix of order  $n$  and suppose that  $\text{per}(A) = 2$ . Show that there exists an integer  $k \geq 2$  and a square submatrix  $B$  of  $A$  whose lines can be permuted to obtain a  $(0,1)$ -matrix with 1's exactly in the positions  $(1,1), (2,2), \dots, (k,k), (1,2), \dots, (k-1,k), (k,1)$ .
2. Deduce that the matrix  $B$  of Smith in (1.22) is a positive definite matrix.
3. Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  distinct positive integers. Let  $A = [a_{ij}]$  be the *greatest common divisor matrix* for  $X$  defined by  $a_{ij} = (x_i, x_j)$ ,  $(i, j) = 1, 2, \dots, n$ . If  $X$  is *factor closed* in the sense that each positive integral divisor of an element in  $X$  is also in  $X$ , then generalize the argument in the proof of Theorem 1.4.5 to evaluate the determinant of  $A$ . Prove that the matrix  $A$  is positive definite for all  $X$  (Beslin and Ligh[1989]).

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## Matrices and Graphs

### 2.1 Basic Concepts

A graph  $G$  (*simple graph*) consists of a finite set  $V = \{a, b, c, \dots\}$  of elements called *vertices* (*points*) together with a prescribed set  $E$  of *unordered* pairs of *distinct* vertices of  $V$ . (The set  $E$  is necessarily finite.) The number  $n$  of elements in the finite set  $V$  is called the *order* of the graph  $G$ . Every unordered pair  $\alpha$  of vertices  $a$  and  $b$  in  $E$  is called an *edge* (*line*) of the graph  $G$ , written

$$\alpha = \{a, b\} = \{b, a\}.$$

We call  $a$  and  $b$  the *endpoints* of  $\alpha$ . Two vertices on the same edge or two distinct edges with a common vertex are *adjacent*. Also, an edge and a vertex are *incident* with one another if the vertex is contained in the edge. Those vertices incident with no edge are *isolated*. A *complete graph* is one in which all possible pairs of vertices are edges. Let  $G$  be a graph and let  $K$  be the complete graph with the same vertex set  $V$ . Then the *complement*  $\bar{G}$  of  $G$  is the graph with vertex set  $V$  and with edge set equal to the set of edges of  $K$  minus those of  $G$ .

A *subgraph* of a graph  $G$  consists of a subset  $V'$  of  $V$  and a subset  $E'$  of  $E$  that themselves form a graph. If  $E'$  contains all edges of  $G$  both of whose endpoints belong to  $V'$ , then the subgraph is called an *induced subgraph* and is denoted by  $G(V')$ . A *spanning subgraph* of  $G$  has the same vertex set as  $G$ . Two graphs  $G$  and  $G'$  are *isomorphic* provided there exists a 1-1 correspondence between their vertex sets that preserves adjacency. Two complete graphs with the same order are isomorphic, and we denote a complete graph of order  $n$  by  $K_n$ .

If the definition of a graph is altered to allow a pair of vertices to form more than one distinct edge, then the structure is called a *multigraph*. Its

edges are called *multiedges (multilines)* and the number of distinct edges of the form  $\{a, b\}$  is called the *multiplicity*  $m\{a, b\}$  of the edge  $\{a, b\}$ . The further generalization by allowing *loops*, edges of the form  $\{a, a\}$  making a vertex adjacent to itself, results in a *general graph*. For both multigraphs and general graphs we require that the edge sets be finite. Terms such as *order*, *endpoints*, *adjacent*, *incident*, *isolated*, etc. carry over directly to multigraphs and general graphs.

Let  $G$  be a multigraph. Then the *degree (valency)* of a vertex in  $G$  is the number of edges incident with the vertex. Since each edge of  $G$  has two distinct endpoints, the sum of the degrees of the vertices of  $G$  is twice the number of its edges. The graph  $G$  is *regular* if all vertices have the same degree. If there are precisely  $k$  edges incident with each vertex of a graph, then we say that the graph is *regular of degree  $k$* . A regular graph of degree 3 is called *cubic*.

One may ask for the number of graphs of a specified order  $n$ . This number has been determined in a certain sense. But the answer is far from elementary and we refer the reader to Harary and Palmer[1973] for a discussion of a variety of problems dealing with graphical enumeration.

### Exercises

1. Prove there are as many graphs of order  $n$  with  $k$  edges as there are with  $\binom{n}{2} - k$  edges. Determine the number of graphs of order at most 5.
2. Prove that a graph always has two distinct vertices with the same degree. Show by example that this need not hold for multigraphs.
3. Prove that a cubic graph has an even number of vertices.

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Let  $G$  denote a general graph of order  $n$  with vertex set

$$V = \{a_1, a_2, \dots, a_n\}.$$

We let  $a_{ij}$  equal the multiplicity  $m\{a_i, a_j\}$  of the edges of the form  $\{a_i, a_j\}$ . This means, of course, that  $a_{ij} = 0$  if there are no edges of the form  $\{a_i, a_j\}$ . Also,  $m\{a_i, a_i\}$  equals the number of loops at vertex  $a_i$ . The resulting matrix

$$A = [a_{ij}], \quad (i, j, = 1, 2, \dots, n)$$

of order  $n$  is called the *adjacency matrix* of  $G$ . The matrix  $A$  characterizes  $G$ .

We note that  $A$  is a symmetric matrix with nonnegative integral elements. The trace of  $A$  denotes the number of loops in  $G$ . If  $G$  is a multigraph, then the trace of  $A$  is zero and the sum of line  $i$  of  $A$  equals the degree of vertex  $a_i$ . If  $G$  is a graph, then  $A$  is a symmetric  $(0,1)$ -matrix of trace zero.

The concept of graph isomorphism has a direct interpretation in terms of the adjacency matrix of the graph. Thus let  $G$  and  $G'$  denote two general graphs of order  $n$  and let the adjacency matrices of these graphs be denoted by  $A$  and  $A'$ , respectively. Then the general graphs  $G$  and  $G'$  are isomorphic if and only if  $A$  is transformable into  $A'$  by simultaneous permutations of the lines of  $A$ . Thus  $G$  and  $G'$  are isomorphic if and only if there exists a permutation matrix  $P$  of order  $n$  such that

$$PAP^T = A'.$$

Let  $G$  be a general graph. A sequence of  $m$  successively adjacent edges

$$\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{m-1}, a_m\}, (m > 0)$$

is called a *walk of length  $m$* , and is also denoted by

$$a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_{m-1} \rightarrow a_m$$

and by

$$a_0, a_1, a_2, \dots, a_{m-1}, a_m.$$

The vertices  $a_0$  and  $a_m$  are the *endpoints* of the walk. The walk is *closed* or *open* according as  $a_0 = a_m$  or  $a_0 \neq a_m$ . A walk with distinct edges is called a *trail*. A walk with distinct edges and in addition distinct vertices (except, possibly,  $a_0 = a_m$ ) is called a *chain*. A closed chain is called a *cycle*. Notice that in a graph a cycle must contain at least 3 edges. But in a general graph a loop or a pair of multiple edges form a cycle.

Let us now form

$$A^2 = \left[ \sum_{t=1}^n a_{it}a_{tj} \right], \quad (i, j = 1, 2, \dots, n). \quad (2.1)$$

Then (2.1) implies that the element in the  $(i, j)$  position of  $A^2$  equals the number of walks of length 2 with  $a_i$  and  $a_j$  as endpoints. In general, the element in the  $(i, j)$  position of  $A^k$  equals the number of walks of length  $k$  with  $a_i$  and  $a_j$  as endpoints. The numbers for closed walks appear on the main diagonal of  $A^k$ .

Let  $G$  be the complete graph  $K_n$  of order  $n$ . We determine the number of walks of length  $k$  in  $K_n$  with  $a_i$  and  $a_j$  as endpoints. The adjacency matrix of  $K_n$  is

$$A = J - I.$$

We know that  $J^e = n^{e-1}J$  so that

$$A^k = \left[ n^{k-1} - \binom{k}{1} n^{k-2} + \binom{k}{2} n^{k-3} - \dots + (-1)^{k-1} \binom{k}{k-1} \right] J + (-1)^k I.$$

But

$$(n-1)^k = n^k - \binom{k}{1} n^{k-1} + \binom{k}{2} n^{k-2} - \dots + (-1)^{k-1} \binom{k}{k-1} n + (-1)^k$$

and hence we have

$$A^k = \left( \frac{(n-1)^k - (-1)^k}{n} \right) J + (-1)^k I.$$

We return to the general graph  $G$  and its adjacency matrix  $A$ . The polynomial

$$f(\lambda) = \det(\lambda I - A)$$

is called the *characteristic polynomial* of  $G$ . The collection of the  $n$  eigenvalues of  $A$  is called the *spectrum* of  $G$ . Since  $A$  is symmetric the spectrum of  $G$  consists of  $n$  real numbers.

Suppose that  $G$  and  $G'$  are isomorphic general graphs. Then we have noted that there exists a permutation matrix  $P$  such that the adjacency matrices  $A$  and  $A'$  of  $G$  and  $G'$ , respectively, satisfy

$$PAP^T = A'.$$

But the transpose of a permutation matrix is equal to its inverse. Thus  $A$  and  $A'$  are similar matrices and hence  $G$  and  $G'$  have the same spectrum. Two nonisomorphic general graphs  $G$  and  $G'$  with the same spectral graph are called *cospectral*. We exhibit in Figure 2.1 two pairs of cospectral graphs of orders 5 and 6 with characteristic polynomials  $f(\lambda) = (\lambda - 2)(\lambda + 2)\lambda^3$  and  $f(\lambda) = (\lambda - 1)(\lambda + 1)^2(\lambda^3 - \lambda^2 - 5\lambda + 1)$ , respectively.

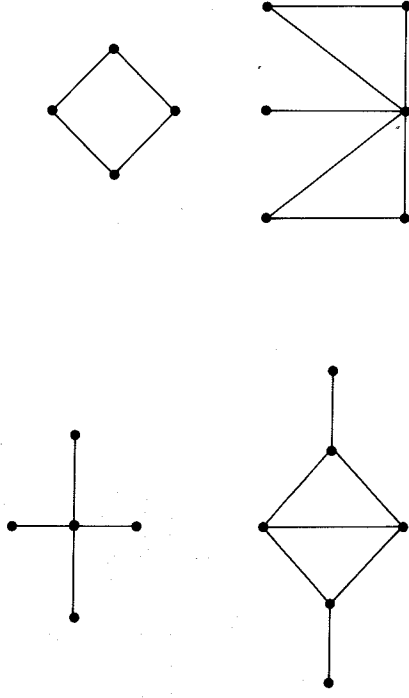


Figure 2.1. Two pairs of cospectral graphs.

A general graph  $G$  is *connected* provided that every pair of vertices  $a$  and  $b$  is joined by a walk with  $a$  and  $b$  as endpoints. A vertex is regarded as trivially connected to itself. Otherwise, the general graph is *disconnected*. Connectivity between vertices is reflexive, symmetric, and transitive. Hence connectivity defines an equivalence relation on the vertices of  $G$  and yields a partition

$$V_1 \cup V_2 \cup \dots \cup V_t,$$

of the vertices of  $G$ . The induced subgraphs  $G(V_1), G(V_2), \dots, G(V_t)$  of  $G$  formed by taking the vertices in an equivalence class and the edges incident to them are called the *connected components* of  $G$ . For most problems concerning  $G$  it suffices to study only the connected components of  $G$ .

Connectivity has a direct interpretation in terms of the adjacency matrix  $A$  of  $G$ . Thus we may simultaneously permute the lines of  $A$  so that  $A$  is transformed into a direct sum of the form

$$A_1 \oplus A_2 \oplus \dots \oplus A_t,$$

where  $A_i$  is the adjacency matrix of the connected component  $G(V_i)$ ,  $(i = 1, 2, \dots, t)$ .

Let  $G$  be a connected general graph. The length of the shortest walk between two vertices  $a$  and  $b$  is the *distance*  $d(a, b)$  between  $a$  and  $b$  in  $G$ . A vertex is regarded as distance 0 from itself. The maximum value of the distance function over all pairs of vertices is called the *diameter* of  $G$ .

**Theorem 2.2.1.** *Let  $G$  be a connected general graph of diameter  $d$ . Then  $G$  has at least  $d + 1$  distinct eigenvalues in its spectrum.*

*Proof.* Let  $a$  and  $b$  be vertices with  $d(a, b) = d$  and let

$$a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_d$$

be a walk of length equal to the diameter  $d$ . Then for each  $i = 1, 2, \dots, d$  there is at least one walk of length  $i$  and no shorter walk that joins  $a_0$  to  $a_i$ . Thus  $A^i$  has a nonzero entry in the position determined by  $a_0$  and  $a_i$ , whereas  $I, A, A^2, \dots, A^{i-1}$  each have zeros in this position. We conclude that  $A^i$  is not a linear combination of  $I, A, A^2, \dots, A^{i-1}$ . Hence the minimum polynomial of  $A$  is of degree at least  $d + 1$ . But since  $A$  is a real symmetric matrix it is similar to a diagonal matrix and consequently the zeros of its minimal polynomial are distinct.  $\square$

### Exercises

1. Prove that the complement of a disconnected graph is a connected graph.
2. Determine the spectrum of the complete graph  $K_n$  of order  $n$ .
3. Show that  $k$  is an eigenvalue of a regular graph of degree  $k$ .
4. Let  $G$  be a graph of order  $n$ . Suppose that  $G$  is regular of degree  $k$  and let  $\lambda_1 = k, \lambda_2, \dots, \lambda_n$  be the spectrum of  $G$ . Prove that the spectrum of the complement of  $G$  is  $n - 1 - k, -1 - \lambda_2, \dots, -1 - \lambda_n$ .
5. Let  $f(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$  be the characteristic polynomial of a graph  $G$  of order  $n$ . Prove that  $c_1$  equals 0,  $c_2$  equals  $-1$  times the number of edges of  $G$ , and  $c_3$  equals  $-2$  times the number of cycles of length 3 of  $G$  (a cycle of length 3 in a graph is sometimes called a *triangle*).
6. Let  $K_{1,n-1}$  be the graph of order  $n$  whose vertices have degrees  $n - 1, 1, \dots, 1$ , respectively. ( $K_{1,n-1}$  is the *star* of order  $n$ .) Prove that the spectrum of  $K_{1,n-1}$  is  $\pm\sqrt{n-1}, 0, \dots, 0$ .
7. Prove that there does not exist a connected graph which is cospectral with the star  $K_{1,n-1}$ .
8. Let  $G$  be a connected graph of order  $n$  which is regular of degree 2. The edges of  $G$  thus form a cycle of length  $n$ , and  $G$  is sometimes called a *cycle graph* of order  $n$ . Determine the spectrum of  $G$ .

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## 2.3 The Incidence Matrix of a Graph

Let  $G$  be a general graph of order  $n$  with vertices  $a_1, a_2, \dots, a_n$  and edges  $\alpha_1, \alpha_2, \dots, \alpha_m$ . We set  $a_{ij} = 1$  if vertex  $a_j$  is on edge  $\alpha_i$  and we set  $a_{ij} = 0$  otherwise. The resulting  $(0,1)$ -matrix

$$A = [a_{ij}], \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$$

of size  $m$  by  $n$  is called the *incidence matrix* of  $G$ . The matrix  $A$  is in fact the conventional incidence matrix in which the edges are regarded as subsets of vertices. Each row of  $A$  contains at least one 1 and not more than two 1's. The rows with a single 1 in  $A$  correspond to the edges in  $G$  that are loops. Identical rows in  $A$  correspond to multiple edges in  $G$ .

The simple row structure of the matrix  $A$  is misleading because  $A$  describes the full complexity of the general graph  $G$ . For example, there is no computationally effective procedure known for the determination of the minimal number of columns in  $A$  with the property that these columns of  $A$  collectively contain at least one 1 in each of the  $m$  rows of  $A$ . In terms of  $G$  this quantity is the minimal number of vertices in  $G$  that touch all edges.

The incidence matrix and the adjacency matrix of a multigraph are related in the following way.

**Theorem 2.3.1.** Let  $G$  be a multigraph of order  $n$ . Let  $A$  be the incidence matrix of  $G$  and let  $B$  be the adjacency matrix of  $G$ . Then

$$A^T A = D + B,$$

where  $D$  is a diagonal matrix of order  $n$  whose diagonal entry  $d_i$  is the degree of the vertex  $a_i$  of  $G$ , ( $i = 1, 2, \dots, n$ ).

*Proof.* The inner product of columns  $i$  and  $j$  of  $A$  ( $i \neq j$ ) equals the multiplicity  $m\{a_i, a_j\}$  of the edge  $\{a_i, a_j\}$ . The inner product of column  $i$  of  $A$  with itself equals the degree of the vertex  $a_i$ .  $\square$

Now let  $G$  denote a graph of order  $n$  with vertices  $a_1, a_2, \dots, a_n$  and edges  $\alpha_1, \alpha_2, \dots, \alpha_m$ . We assign to each of the edges of  $G$  one of the two possible orientations and thereby transform  $G$  into a graph in which each of the edges of  $G$  is assigned a direction. We set  $a_{ij} = 1$  if  $a_j$  is the "initial" vertex of  $\alpha_i$ , we set  $a_{ij} = -1$  if  $a_j$  is the "terminal" vertex of  $\alpha_i$  and we set  $a_{ij} = 0$  if  $a_j$  is not an endpoint of  $\alpha_i$ . The resulting  $(0, 1, -1)$ -matrix

$$A = [a_{ij}], \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$$

of size  $m$  by  $n$  is called the *oriented incidence matrix* of  $G$ . Each row of  $A$  contains exactly two nonzero entries, one of which is 1 and the other  $-1$ .

We note that in the notation of Theorem 2.3.1 the oriented incidence matrix satisfies

$$A^T A = D - B. \quad (2.2)$$

The matrix  $A^T A$  in (2.2) is called the *Laplacian matrix* (also called the *admittance matrix*) of  $G$ . It follows from (2.2) that the Laplacian matrix is independent of the particular orientation assigned to  $G$ . The Laplacian matrix will be discussed further in section 2.5.

The oriented incidence matrix is used to determine the number of connected components of  $G$ .

**Theorem 2.3.2.** *Let  $G$  be a graph of order  $n$  and let  $t$  denote the number of connected components of  $G$ . Then the oriented incidence matrix  $A$  of  $G$  has rank  $n - t$ . In fact, each matrix obtained from  $A$  by deleting  $t$  columns, one corresponding to a vertex of each component, has rank  $n - t$ . A submatrix  $A'$  of  $A$  of order  $n - 1$  has rank  $n - t$  if and only if the spanning subgraph  $G'$  of  $G$  whose edges are those corresponding to the rows of  $A'$  has  $t$  connected components.*

*Proof.* Let the connected components of  $G$  be denoted by

$$G(V_1), G(V_2), \dots, G(V_t).$$

Then we may label the vertices and edges of  $G$  so that the oriented incidence matrix  $A$  is a direct sum of the form

$$A_1 \oplus A_2 \oplus \dots \oplus A_t,$$

where  $A_i$  displays the vertices and edges in  $G(V_i)$ , ( $i = 1, 2, \dots, t$ ). Let  $G(V_i)$  contain  $n_i$  vertices. We prove that the rank of  $A_i$  equals  $n_i - 1$ . The conclusion then follows by addition.

Let  $\beta_j$  denote the column of  $A_i$  corresponding to the vertex  $a_j$  of  $G(V_i)$ . Since each row of  $A_i$  contains exactly one 1 and one  $-1$ , it follows that the sum of the columns of  $A_i$  is the zero vector. Hence the rank of  $A_i$  is at most  $n_i - 1$ . Suppose then that we have a linear relation  $\sum b_j \beta_j = 0$ , where the summation is over all columns of  $A_i$  and not all the coefficients are zero. Let us suppose that column  $\beta_k$  has  $b_k \neq 0$ . This column has nonzero entries in those rows corresponding to the edges incident with  $a_k$ . For each such row there is just one other column  $\beta_l$  with a nonzero entry in that row. In order for the dependency to hold we must have  $b_k = b_l$ . Hence if  $b_k \neq 0$ , then  $b_l = b_k$  for all vertices  $a_l$  adjacent to  $a_k$ . Since  $G(V_i)$  is connected it follows that all of the coefficients  $b_j$  are equal and the linear relation is merely a multiple of our earlier relation  $\sum \beta_j = 0$ . Hence the rank of  $G(V_i)$  is  $n_i - 1$ , and deleting any column of  $A_i$  results in a matrix of rank  $n_i - 1$ . Finally we observe that the last conclusion of the theorem follows by applying the earlier conclusions to  $G'$ .  $\square$

A matrix  $A$  with integral elements is *totally unimodular* if every square submatrix of  $A$  has determinant 0, 1, or  $-1$ . It follows at once that a totally unimodular matrix is a  $(0, 1, -1)$ -matrix.

The following theorem is due to Hoffman and Kruskal[1956].

**Theorem 2.3.3.** *Let  $A$  be an  $m$  by  $n$  matrix whose rows are partitioned into two disjoint sets  $B$  and  $C$  and suppose that the following four properties hold:*

- (i) *Every entry of  $A$  is 0, 1, or  $-1$ .*
- (ii) *Every column of  $A$  contains at most two nonzero entries.*
- (iii) *If two nonzero entries in a column of  $A$  have the same sign, then the row of one is in  $B$  and the row of the other is in  $C$ .*
- (iv) *If two nonzero entries in a column of  $A$  have opposite signs, then the rows of both are in  $B$  or in  $C$ .*

*Then the matrix  $A$  is totally unimodular.*

*Proof.* An arbitrary submatrix of  $A$  also satisfies the hypothesis of the theorem. Hence it suffices to prove that an arbitrary square matrix  $A$  satisfying the hypotheses of the theorem has  $\det(A)$  equal to 0, 1, or  $-1$ . The proof is by induction on  $n$ . For  $n = 1$  the theorem follows trivially from (i). Suppose that every column of  $A$  has two nonzero entries. Then the sum of the rows in  $B$  equals the sum of the rows in  $C$  and  $\det(A) = 0$ . This assertion is also valid in case  $B = \emptyset$  or  $C = \emptyset$ . Also if some column of  $A$  has all 0's, then  $\det(A) = 0$ . Hence we are left with the case in which some column of  $A$  has exactly one nonzero entry. We expand  $\det(A)$  by this column and apply the induction hypothesis.  $\square$

The preceding result implies the following theorem of Poincaré[1901].

**Corollary 2.3.4.** *The oriented incidence matrix  $A$  of a graph  $G$  is totally unimodular.*

*Proof.* We apply Theorem 2.3.3 to the matrix  $A^T$  with  $C = \emptyset$ .  $\square$

A square  $(0, 1, -1)$ -matrix is *Eulerian* provided all line sums are even integers. Camion[1965] (see also Padberg[1976]) has established the following theorem giving a necessary and sufficient condition for total unimodularity which we state without proof.

**Theorem 2.3.5.** *A  $(0, 1, -1)$ -matrix  $A$  of size  $m$  by  $n$  is totally unimodular if and only if the sum of the elements in each Eulerian submatrix is a multiple of 4.*

Totally unimodular matrices are intimately related to a special class of *matroids* (see White[1986]) called *unimodular matroids* (also called *regular matroids*). These are matroids which can be coordinatized by a matrix  $A$  over the rational field for which  $A$  is totally unimodular. Unimodular

matroids were characterized by Tutte[1958] (see also Gerards[1989]) in a very striking theorem. Another striking characterization of unimodular matroids was obtained by Seymour[1980]. The characterizations of Tutte and of Seymour are in terms of the linear dependence structure of the columns of the matrix  $A$ .

A tree is a connected graph that contains no cycle. We will assume a familiarity with a few of the most elementary properties of trees (Brualdi [1977] or Wilson[1972]). Let  $T$  be a graph of order  $n$ . Then the following statements are equivalent: (1)  $T$  is a tree; (2)  $T$  contains no cycles and has exactly  $n - 1$  edges; (3)  $T$  is connected and has exactly  $n - 1$  edges; (4) each pair of distinct vertices of  $T$  is joined by exactly one chain.

Let  $T$  be a tree of order  $n$  with vertices  $a_1, a_2, \dots, a_n$  and edges  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ . We suppose that the edges of  $T$  have been oriented. Let  $(s_i, t_i)$ ,  $(i = 1, 2, \dots, l)$  be  $l$  ordered pairs of vertices of  $T$ . We set  $m_{ij} = 1$  if the unique chain  $\gamma$  in  $T$  joining  $s_i$  and  $t_j$  uses the edge  $\alpha_j$  in its assigned direction, we set  $m_{ij} = -1$  if the chain  $\gamma$  uses the edge  $\alpha_j$  in the direction opposite to its assigned direction, and we set  $m_{ij} = 0$  if  $\gamma$  does not use the edge  $\alpha_j$ . The resulting  $(0, 1, -1)$ -matrix

$$M = [m_{ij}], \quad (i = 1, 2, \dots, l; j = 1, 2, \dots, n - 1)$$

of size  $l$  by  $n - 1$  is called a *network matrix* (Tutte[1965]). If we delete the column of the network matrix  $A$  corresponding to the arc  $\alpha_k$  of  $G$ , the resulting matrix is a network matrix for the tree obtained from  $T$  by contracting the edge  $\alpha_k$ , that is by deleting the arc  $\alpha_k$  and identifying its two endpoints. It follows that submatrices of network matrices are also network matrices.

**Theorem 2.3.6.** *A network matrix  $M$  corresponding to the oriented tree  $T$  is a totally unimodular matrix.*

*Proof.* We continue with the notation in the preceding paragraph. Let  $G$  be the graph with vertices  $a_1, a_2, \dots, a_n$  and edges  $\{s_i, t_i\}$ ,  $(i = 1, 2, \dots, l)$ . We orient each edge  $\{s_i, t_i\}$  from  $s_i$  to  $t_i$ . Let  $A$  be the  $l$  by  $n$  oriented incidence matrix of  $G$ , and let  $B$  be the  $n - 1$  by  $n$  oriented incidence matrix of  $T$ . Let  $A'$  and  $B'$  result from  $A$  and  $B$ , respectively, by deleting the last column (the column corresponding to vertex  $a_n$  in each case). From the definitions of the matrices involved we obtain the relation  $MB = A$ . Hence  $MB' = A'$ , and since by Theorem 2.3.2  $B'$  is invertible, the relation  $M = A'B'^{-1}$  holds. By Theorem 2.3.3 the matrix

$$\begin{bmatrix} I_{n-1} \\ B' \\ A' \end{bmatrix}$$

is totally unimodular. It follows that the matrix

$$\begin{bmatrix} I_{n-1} \\ B' \\ A' \end{bmatrix} B'^{-1} = \begin{bmatrix} B'^{-1} \\ I_{n-1} \\ A'B'^{-1} \end{bmatrix}$$

has the property that all of its submatrices of order  $n - 1$  have determinants equal to one of  $0, 1$  and  $-1$ . This implies that the matrix  $M = A'B'^{-1}$  is totally unimodular.  $\square$

Seymour's characterization of unimodular matroids can be restated in matrix terms. In this characterization the totally unimodular matrices

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \tag{2.3}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2.4}$$

have an exceptional role.

The following theorem of Seymour[1982] asserts that a totally unimodular matrix which is not a network matrix, the transpose of a network matrix, or one of the two exceptional matrices above admits a "diagonal decomposition" into smaller totally unimodular matrices.

**Theorem 2.3.7.** *Let  $A$  be a totally unimodular matrix. Then one of the following properties holds:*

- (i)  $A$  is a network matrix or the transpose of a network matrix;
- (ii)  $A$  can be obtained from one of the two exceptional matrices above by line permutations and by the multiplication of some of their lines by  $-1$ ;
- (iii) The lines of  $A$  can be permuted to obtain a matrix

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where  $B_{11}$  and  $B_{22}$  are totally unimodular matrices, and either

- (a)  $\text{rank}(B_{12}) + \text{rank}(B_{21}) = 0$   
and  $B_{11}$  and  $B_{22}$  each have at least one line, or

- (b)  $\text{rank}(B_{12}) + \text{rank}(B_{21}) = 1$   
 and  $B_{11}$  and  $B_{22}$  each have at least two lines, or  
 (c)  $\text{rank}(B_{12}) + \text{rank}(B_{21}) = 2$   
 and  $B_{11}$  and  $B_{22}$  each have at least six lines.

The above theorem implies that it is possible to construct the entire class of totally unimodular network matrices from the class of network matrices and the two exceptional matrices (2.3) and (2.4). Neither of these exceptional matrices is a network matrix or the transpose of a network matrix. If the matrix  $A$  satisfies (iii) above, then it does not necessarily follow that  $A$  is totally unimodular. However, a result of Brylawski[1975] gives a list of conditions such that a matrix  $A$  which satisfies (iii) is totally unimodular and such that every totally unimodular matrix can be constructed in the manner of (iii) starting from the network matrices, the transposes of the network matrices and the two exceptional matrices. These conditions are difficult to state and we refer the reader to the original paper by Brylawski. A consequence of Seymour's characterizations of unimodular matroids and totally unimodular matrices is the existence of an algorithm to determine whether a  $(0, 1, -1)$ -matrix is totally unimodular whose number of steps is bounded by a polynomial function in the number of lines of the matrix (see Schrijver[1986]).

### Exercises

1. Verify that the matrices (2.3) and (2.4) are totally unimodular.
2. Prove that each nonsingular submatrix of a totally unimodular matrix has an integral inverse.
3. Let  $A$  be a totally unimodular matrix of size  $m$  by  $n$ . Let  $b$  be a matrix of size  $m$  by 1 each of whose elements is an integer. Prove that the consistent equation  $Ax = b$  has an integral solution.
4. Let  $A$  be a totally unimodular matrix and let  $B$  be a nonsingular submatrix of  $A$  of order  $k$ . Prove that for each nonzero  $(0, 1, -1)$ -vector  $y$  of size  $k$ , the greatest common divisor of the elements of the vector  $yB$  equals 1. [Indeed Chandrasekaran has shown that this property characterizes totally unimodular matrices (see Schrijver[1986]).]
5. Let  $A$  be a nonsingular  $(0, 1, -1)$ -matrix and suppose that  $|\det(A)| \neq 1$ . Prove that  $A$  has a square submatrix  $B$  with  $|\det(B)| = 2$ .

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## 2.4 Line Graphs

Let  $G$  denote a graph of order  $n$  on  $m$  edges. The line graph  $L(G)$  of  $G$  is the graph whose vertices are the edges of  $G$  and two vertices of  $L(G)$  are adjacent if and only if the corresponding edges of  $G$  have a vertex in common. The line graph  $L(G)$  is of order  $m$ .

**Theorem 2.4.1.** Let  $A$  be the incidence matrix of a graph  $G$  on  $m$  edges and let  $B_L$  be the adjacency matrix of the line graph  $L(G)$ . Then

$$AA^T = 2I_m + B_L. \quad (2.5)$$

*Proof.* For  $i \neq j$  the entry in the  $(i, j)$  position of  $B_L$  is 1 if the edges  $\alpha_i$  and  $\alpha_j$  of  $G$  have a vertex in common and 0 otherwise. But the same conclusion holds for the entry in the  $(i, j)$  position of  $AA^T$ . The main diagonal elements are as indicated.  $\square$

Theorem 2.4.1 implies a severe restriction on the spectrum of a line graph.

**Theorem 2.4.2.** If  $\lambda$  is an eigenvalue of the line graph  $L(G)$ , then  $\lambda \geq -2$ . If  $G$  has more edges than vertices, then  $\lambda = -2$  is an eigenvalue of  $L(G)$ .

*Proof.* The symmetric matrix  $AA^T$  is positive semidefinite and hence its eigenvalues are nonnegative. But if  $\alpha$  is an eigenvector of  $B_L$  associated with the eigenvalue  $\lambda$ , then (2.5) implies

$$AA^T \alpha = (2 + \lambda)\alpha$$

so that  $\lambda \geq -2$ . If  $G$  has more edges than vertices, then  $AA^T$  is singular and hence 0 is an eigenvalue of  $AA^T$ .  $\square$

If the graph  $G$  is regular of degree  $k$ , then the number of its edges is  $m = nk/2$  and the line graph  $L(G)$  is regular of degree  $2(k-1)$ . The following theorem of Sachs [1967] shows that in this case the characteristic polynomials of  $G$  and  $L(G)$  are related in an elementary way.

**Theorem 2.4.3.** *Let  $G$  be a graph of order  $n$  which is regular of degree  $k$  on  $m$  edges. Let  $f(\lambda)$  and  $g(\lambda)$  be the characteristic polynomials of  $G$  and  $L(G)$ , respectively. Then*

$$g(\lambda) = (\lambda + 2)^{m-n} f(\lambda + 2 - k).$$

*Proof.* We recall that two matrix products of the form  $XY$  and  $YX$  have the same collection of eigenvalues apart from zero eigenvalues. Thus the incidence matrix  $A$  of  $G$  satisfies

$$\det(\lambda I_m - AA^T) = \lambda^{m-n} \det(\lambda I_n - A^T A). \quad (2.6)$$

We let  $B$  and  $B_L$  denote the adjacency matrices of  $G$  and  $L(G)$ , respectively. Then using Theorems 2.3.1 and 2.4.1, we obtain

$$\begin{aligned} \det(\lambda I_m - B_L) &= \det((\lambda + 2)I_m - AA^T) \\ &= (\lambda + 2)^{m-n} \det((\lambda + 2)I_n - A^T A) \\ &= (\lambda + 2)^{m-n} \det((\lambda + 2 - k)I_n - B). \end{aligned} \quad \square$$

The study of the spectral properties of line graphs was initiated by A.J. Hoffman and extensively investigated by him and his associates over a period of many years. The condition  $\lambda \geq -2$  of Theorem 2.4.2 imposes severe restrictions on the spectrum of  $L(G)$ . But graphs other than line graphs exist that also satisfy this requirement. Generalized line graphs were introduced by Hoffman [1970, 1977] as a class of graphs more general than line graphs that satisfy  $\lambda \geq -2$ . We briefly discuss these contributions.

Let  $t$  be a nonnegative integer. The *cocktail party graph*  $CP(t)$  of order  $2t$  is the graph with vertices  $b_1, b_2, \dots, b_{2t}$  in which each pair of distinct vertices form an edge with the exception of the pairs  $\{b_1, b_2\}, \{b_3, b_4\}, \dots, \{b_{2t-1}, b_{2t}\}$ . The cocktail party graph  $CP(t)$  can be obtained from the complete graph  $K_{2t}$  of order  $2t$  by deleting  $t$  edges no two of which are adjacent. If  $t = 0$ , then the cocktail party graph is a graph with no vertices. Now let  $G$  be a graph of order  $n$  with vertices  $a_1, a_2, \dots, a_n$ , and let  $k_1, k_2, \dots, k_n$  be an  $n$ -tuple of nonnegative integers. The *generalized line graph*  $L(G; k_1, k_2, \dots, k_n)$  is the graph of order  $n + 2(k_1 + k_2 + \dots + k_n)$  defined as follows. We begin with the line graph  $L(G)$  and disjoint cocktail party graphs  $CP(k_1), CP(k_2), \dots, CP(k_n)$ . Each vertex of  $L(G)$  is an

edge  $\alpha = \{a_i, a_j\}$  of  $G$  and we put edges between  $\alpha$  and each vertex of  $CP(k_i)$  and of  $CP(k_j)$ . If  $\lambda$  is an eigenvalue of a generalized line graph, then  $\lambda \geq -2$ .

**Theorem 2.4.4.** *If  $G$  is a connected graph of order greater than 36 for which each eigenvalue  $\lambda$  satisfies the condition  $\lambda \geq -2$ , then  $G$  is a generalized line graph.*

The following theorem of Hoffman and Ray-Chaudhuri characterizes regular graphs satisfying  $\lambda \geq -2$ .

**Theorem 2.4.5.** *If  $G$  is a regular connected graph of order greater than 28 for which each eigenvalue  $\lambda$  satisfies the condition  $\lambda \geq -2$ , then  $G$  is either a line graph or a cocktail party graph.*

Finally, we mention the important paper by Cameron, Goethals, Seidel and Shult [1976] in which the above two theorems are obtained by appealing to the classical root systems.

### Exercises

- Determine the spectrum of the line graph  $L(K_n)$ .
- The complement of the line graph of  $K_5$  is a cubic graph of order 10 and is known as the *Petersen graph*. Determine the spectrum of the Petersen graph.
- Find an example of a graph of order 4 which is not isomorphic to the line graph of any graph. Deduce that the Petersen graph is not isomorphic to a line graph.
- Show that the spectrum of the cocktail party graph  $CP(t)$  of order  $2t - 2$ , 0 (with multiplicity  $t$ ), and  $-2$  (with multiplicity  $t - 1$ ).
- Let  $B$  be the adjacency matrix of a generalized line graph  $G$ . Determine a matrix  $N$  such that  $NN^T = 2I + B$ , and then deduce that  $\lambda \geq -2$  for each eigenvalue  $\lambda$  of  $G$ .
- Let  $A$  be the incidence matrix of a tree of order  $n$ . Prove that the rank of  $A$  equals  $n - 1$ .
- Let  $G$  be a connected graph of order  $n$  on  $n$  edges and let  $A$  be the incidence matrix of  $G$ . The graph  $G$  has a unique cycle  $\gamma$ . Prove that  $A$  has rank  $n$  if  $\gamma$  has odd length and that  $A$  has rank  $n - 1$  if  $\gamma$  has even length. Deduce that the incidence matrix of a connected graph of order  $n$  has rank  $n$  if it has an odd length cycle and has rank  $n - 1$  otherwise.
- Let  $G$  be a connected graph of order  $n$ . Prove that the multiplicity of 0 as an eigenvalue of the line graph  $L(G)$  equals  $m - n$  if  $G$  has an odd length cycle and equals  $m - n + 1$  otherwise.

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## 2.5 The Laplacian Matrix of a Graph

Let  $G$  denote a graph of order  $n$  with vertices  $a_1, a_2, \dots, a_n$  and edges  $\alpha_1, \alpha_2, \dots, \alpha_m$ . Let  $A$  be the oriented incidence matrix of  $G$  of size  $m$  by  $n$ , and let  $B$  be the adjacency matrix of  $G$ . Recall that the Laplacian matrix of  $G$  is the matrix of order  $n$

$$F = A^T A = D - B$$

where  $D$  is a diagonal matrix of order  $n$  whose diagonal entry  $d_i$  is the degree of the vertex  $a_i$  of  $G$ , ( $i = 1, 2, \dots, n$ ). By Theorem 2.3.2 the matrix  $A$ , and hence the Laplacian matrix  $F$  has rank at most equal to  $n - 1$ . Thus the matrix  $F$  is a singular matrix. A *spanning tree*  $T$  of  $G$  is a spanning subgraph of  $G$  which forms a tree. Every connected graph contains a spanning tree. Let  $U$  be a subset of the edges of  $G$ . Then we denote by  $\langle U \rangle$  the subgraph of  $G$  consisting of the edges of  $U$  and all vertices of  $G$  incident with at least one edge of  $U$ . The following lemma is an immediate consequence of Theorem 2.3.2.

**Lemma 2.5.1.** *Let  $U$  be an  $(n-1)$ -subset of edges of the connected graph  $G$  of order  $n$ . Let  $A_U$  denote a submatrix of order  $n-1$  of the oriented incidence matrix  $A$  of  $G$  consisting of the intersection of the  $n-1$  rows of  $A$  corresponding to the edges of  $U$  and any set of  $n-1$  columns of  $A$ . Then  $A_U$  is nonsingular if and only if  $\langle U \rangle$  is a spanning tree of  $G$ .*

The complexity of a graph  $G$  of order  $n$  is the number of spanning trees of  $G$ . We denote the complexity of  $G$  by  $c(G)$ . In case  $G$  is disconnected we have  $c(G) = 0$ .

**Lemma 2.5.2.** *Let  $A$  be the oriented incidence matrix of a graph  $G$  of order  $n$ . Then the adjugate of the Laplacian matrix*

$$F = A^T A = D - B$$

*is a multiple of  $J$ .*

*Proof.* If  $G$  is disconnected, then  $\text{rank}(F) = \text{rank}(A) < n - 1$  and hence  $\text{adj}(F) = O$ .

If  $G$  is connected, then  $\text{rank}(F) = n - 1$ . But since

$$F \text{adj}(F) = \det(F)I = O$$

it follows that each column of  $\text{adj}(F)$  is in the kernel of  $F$ . But this kernel is a one-dimensional space spanned by the vector  $e_n = (1, 1, \dots, 1)^T$ . Hence each column of  $\text{adj}(F)$  is a multiple of  $e_n$ . But  $F = A^T A$  is symmetric and this implies that  $\text{adj}(F)$  is also symmetric. Hence it follows that  $\text{adj}(A)$  is a multiple of  $J$ .  $\square$

We now obtain a classical formula.

**Theorem 2.5.3.** *In the above notation we have*

$$\text{adj}(F) = c(G)J.$$

*Proof.* By Lemma 2.5.2 we need only show that one cofactor of  $F$  is equal to  $c(G)$ . Let  $A_0$  denote the matrix obtained from  $A$  by removing the last column of  $A$ . It follows that  $\det(A_0^T A_0)$  is a cofactor of  $F$ . Now let  $A_U$  denote a submatrix of order  $n-1$  of  $A_0$  whose rows correspond to the edges in an  $(n-1)$ -subset  $U$  of the edges of  $G$ . Then by the Binet-Cauchy theorem we have

$$\det(A_0^T A_0) = \sum \det(A_U^T) \det(A_U),$$

where the summation is over all possible choices of  $U$ . By Lemma 2.5.1 we have that  $A_U$  is nonsingular if and only if  $\langle U \rangle$  is a spanning tree of  $G$ , and in this case by Corollary 2.3.4 we have  $\det(A_U) = \pm 1$ . But  $\det(A_U) = \det(A_U^T)$  so that  $\det(A_0^T A_0) = c(G)$  and the conclusion follows.  $\square$

For the complete graph  $K_n$  of order  $n$  we have  $F = nI - J$ , and an easy calculation (cf. Exercise 1, Sec. 1.3) yields the famous Cayley formula [1889]

$$c(K_n) = n^{n-2}$$

for the number of labeled trees of order  $n$ .

Theorem 2.5.3 may be formulated in an even more elegant form (Tenperley [1964]).

**Theorem 2.5.4.** *The complexity of a graph  $G$  of order  $n$  is given by the formula*

$$c(G) = n^{-2} \det(F + J).$$

*Proof.* We have  $J^2 = nJ$  and  $FJ = O$  so that

$$(F + J)(nI - J) = nF.$$

We take the adjugate of both sides and use Theorem 2.5.3 to obtain

$$n^{n-2} J \operatorname{adj}(F + J) = n^{n-1} c(G) J.$$

We now multiply by  $F + J$  and this gives

$$\det(F + J) J = n^2 c(G) J,$$

as desired.  $\square$

Now let  $(x_1, x_2, \dots, x_n)^T$  be a real  $n$ -vector. Then

$$x^T F x = x^T A^T A x = \sum_{\alpha_i = \{a_i, a_j\}} (x_i - x_j)^2 \quad (2.7)$$

where the summation is over all  $m$  edges  $\alpha_t = \{a_i, a_j\}$  of  $G$ . The matrix  $F$  is a positive semidefinite symmetric matrix. Moreover, 0 is an eigenvalue of  $F$  with corresponding eigenvector  $e_n = (1, 1, \dots, 1)^T$ . Let  $\mu = \mu(G)$  denote the second smallest eigenvalue of  $F$ . In case  $n = 1$  we define  $\mu$  to be 0. We have from Theorem 2.3.2 that  $\mu \geq 0$  with equality if and only if  $G$  is a disconnected graph. Fiedler[1973] defined  $\mu$  to be the algebraic connectivity of the graph  $G$ . The algebraic connectivity of the complete graph  $K_n$  of order  $n$  is  $n$  (cf. Exercise 2, Sec. 2.2).

Let  $U$  be the set of all real  $n$ -tuples  $x$  such that  $x^T x = 1$  and  $x^T e_n = 0$ . From the theory of symmetric matrices (see, e.g., Horn and Johnson[1987]) we obtain the characterization

$$\mu = \min\{x^T F x \mid x \in U\} \quad (2.8)$$

for the algebraic connectivity of the graph  $G$ . It follows from equations (2.7) and (2.8) that if  $G'$  is a spanning subgraph of  $G$  then  $\mu(G') \leq \mu(G)$ . Thus for graphs with the same set of vertices the algebraic connectivity is a nondecreasing function of the edges.

**Theorem 2.5.5.** Let  $G_t$  be a graph of order  $n$  which is obtained from the graph  $G$  by removing the vertex  $a_t$  and all edges incident with  $a_t$ . Then

$$\mu(G_t) \geq \mu(G) - 1.$$

*Proof.* Let  $F_t$  be the Laplacian matrix of  $G_t$ . First suppose that  $a_t$  is adjacent to all other vertices in  $G$ . Then the matrix  $F_t + I$  is a principal submatrix of order  $n - 1$  of the Laplacian matrix  $F$  of  $G$ . By the interlacing inequalities for the eigenvalues of symmetric matrices we have

$$\mu(G_t) + 1 \geq \mu(G).$$

In the general case we use the fact that the algebraic connectivity is a nondecreasing function of the edges.  $\square$

There are two standard ways to measure the extent to which a graph is connected. The vertex connectivity of the graph  $G$  is the smallest number of vertices whose removal from  $G$ , along with each edge incident with at least one of the removed vertices, leaves either a disconnected graph or a graph with a single vertex. Thus if  $G$  is not a complete graph, the vertex connectivity equals  $n - p$  where  $p$  is the largest order of a disconnected induced subgraph of  $G$ . The edge connectivity of  $G$  is the smallest number of edges whose removal from  $G$  leaves a disconnected graph or a graph with one vertex. The edge connectivity is no greater than the minimum degree of the vertices of  $G$ . The vertex and edge connectivities of the complete graph  $K_n$  equal  $n - 1$ .

**Theorem 2.5.6.** Let  $G$  be a graph of order  $n$  which is not complete. The algebraic connectivity  $\mu(G)$ , the vertex connectivity  $v(G)$  and the edge connectivity  $e(G)$  of  $G$  satisfy

$$\mu(G) \leq v(G) \leq e(G).$$

*Proof.* Since  $G$  is not complete there is a disconnected graph  $G^*$  which results from  $G$  by the removal of  $v(G)$  vertices. Then  $\mu(G^*) = 0$  and by repeated application of Theorem 2.5.5 we get  $\mu(G^*) \geq \mu(G) - v(G)$ . Now let  $k = e(G)$  and let  $\alpha_{x_1}, \alpha_{x_2}, \dots, \alpha_{x_k}$  be a set of  $k$  edges whose removal from  $G$  results in a disconnected graph. This disconnected graph has exactly two connected components  $G_1$  and  $G_2$ , and each of the removed edges joins a vertex of  $G_1$  to a vertex of  $G_2$ . Let  $x_j$  be the vertex of  $\alpha_{x_j}$  which belongs to  $G_1$ , ( $j = 1, 2, \dots, k$ ). Notice that the vertices  $x_1, x_2, \dots, x_k$  are not necessarily distinct. If the removal of the vertices  $x_1, x_2, \dots, x_k$  disconnects  $G$ , then  $v(G) \leq k = e(G)$ . Otherwise  $x_1, x_2, \dots, x_k$  are the only vertices of  $G_1$  and it follows that each vertex  $x_i$  has degree at most  $k$ . Hence each vertex  $x_i$  has degree exactly  $k$ . We now delete all the vertices adjacent to  $x_i$  and disconnect the graph  $G$ . Hence  $v(G) \leq k = e(G)$ .  $\square$

We now assume that the graph  $G$  is connected and hence that the algebraic connectivity  $\mu$  of  $G$  is positive. The following theorem of Fiedler[1975] shows that an eigenvector  $x$  of the Laplacian matrix corresponding to its eigenvalue  $\mu$  contains easily accessible information about the graph  $G$ .

**Theorem 2.5.7.** Let  $G$  be a connected graph of order  $n$  with vertices  $a_1, a_2, \dots, a_n$ . Let  $x = (x_1, x_2, \dots, x_n)^T$  be an eigenvector of the Laplacian matrix  $F$  of  $G$  corresponding to the eigenvalue  $\mu$ . Let  $r$  be a nonnegative number, and define

$$V_r = \{a_i \mid x_i + r \geq 0, 1 \leq i \leq n\}.$$

Then the induced subgraph  $G(V_r)$  of  $G$  with vertex set  $V_r$  is connected.

*Proof.* We first prove the theorem under the assumption that  $r = 0$ . In this case  $V_0 = \{a_i | x_i \geq 0, 1 \leq i \leq n\}$ . Suppose to the contrary that the induced subgraph  $G(V_0)$  is disconnected. We simultaneously permute the lines of  $F$  to obtain

$$\begin{bmatrix} F_1 & O & R \\ O & F_2 & S \\ R^T & S^T & F' \end{bmatrix}, \quad (2.9)$$

where  $F_1$  corresponds to a connected component of  $G(V_0)$  and  $F_2$  corresponds to the remaining components of  $G(V_0)$ . We write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x' \end{bmatrix}$$

to conform to the partition of  $F$  in (2.9). The elements of the vectors  $x_1$  and  $x_2$  are nonnegative, and the elements of the vector  $x'$  are negative. The matrix

$$\begin{bmatrix} F_1 & O \\ O & F_2 \end{bmatrix} \quad (2.10)$$

is a principal submatrix of  $F$ . Since 0 is a simple eigenvalue of  $F$ , it follows from the interlacing inequalities for symmetric matrices that the multiplicity of 0 as an eigenvalue of the matrix (2.10) is at most 1. This implies that we may assume that the matrix  $F_1$  is nonsingular. The equation  $Fx = \mu x$  implies that

$$(F_1 - \mu I)x_1 = -Rx'. \quad (2.11)$$

The eigenvalues of  $F_1 - \mu I$  are nonnegative and hence  $F_1 - \mu I$  is a positive semidefinite matrix. The sign patterns of the matrices and vectors involved imply that the elements of the vector  $-Rx'$  are nonpositive. Multiplying equation (2.11) on the left by  $x_1^T$  we obtain

$$x_1^T (F_1 - \mu I)x_1 = -x_1^T R x' \leq 0.$$

Since the matrix  $F_1 - \mu I$  is positive semidefinite, this implies that

$$(F_1 - \mu I)x_1 = 0.$$

By (2.11) we also have  $Rx' = 0$ . Since  $R$  has nonpositive entries and  $x'$  has negative entries, we have  $R = 0$ , contradicting the hypothesis that  $G$  is a connected graph.

For general nonnegative numbers  $r$  we replace the vector  $x$  by  $x + r\mathbf{e}_n$  and then proceed as above.  $\square$

A very similar proof can be given for the following: Let  $r$  be a nonpositive number and define

$$V_r = \{a_i | x_i + r \leq 0, 1 \leq i \leq n\}.$$

Then the induced subgraph  $G(V_r)$  of  $G$  is connected. More general results can be found in Fiedler[1975]. The algebraic connectivity of trees is studied in Grone and Merris[1987] and in Fiedler[1990]. A survey of the eigenvalues of the Laplacian matrix of graphs is given by Mohar[1988]. A more general survey of the Laplacian is given by Grone[1991].

### Exercises

1. Determine the complexity of the Petersen graph.
2. Determine the algebraic connectivity of the star  $K_{1,n-1}$  of order  $n$ .
3. Let  $G$  be a graph of order  $n$  and let  $d$  denote the smallest degree of a vertex of  $G$ . Prove that  $\mu(G) \leq dn/(n-1)$  (Fiedler[1973]).
4. Let  $G$  be a connected graph of order  $n$  which is regular of degree  $k$ . Let the spectrum of  $G$  be  $\lambda_1 = k, \lambda_2, \dots, \lambda_n$ . Use Theorem 2.5.4 to show that the complexity  $c(G)$  satisfies

$$c(G) = \frac{1}{n} \prod_{i=2}^n (k - \lambda_i)$$

(Biggs[1974]).

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