Other Definitions in Graph Theory from Weeks 7–10

March 10 (Day 21)

Modifications of Graphs:

Notation	Mathspeak	Modification in words
$G\backslash v$	delete v	remove v and all edges e incident with v.
$G\backslash e$	delete e	remove e
G/e	contract e	if $e = v_1v_2$, then replace v_1 and v_2 with a "super-vertex" v that is adjacent to all neighbors of v_1 and v_2.

Note: Edge contraction of a graph may result in a multigraph!
Definition: The inverse operation of edge contraction is called *vertex splitting*.
Definition: A graph H is called an *expansion* of G if H is obtained from G by a sequence of vertex splittings.
Definition: *subdividing an edge* e is replacing e with a path of any length.
Definition: A graph H is a *subdivision* of G if H arises by subdividing some of the edges of G.
Note: Successive edge contractions can help revert a subdivision of G back to G.

Definition: A graph H is a *minor* of a graph G if H can be obtained from G via repeated edge deletion and/or edge contraction.
Lemma: If G is not planar, a subdivision of G is not planar.
Lemma: If G contains a nonplanar subgraph, G is not planar.

Theorem: (Kuratowski's Theorem = Book Theorems 9.1.1 AND 9.1.2)
G is nonplanar **if and only if** G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$.
(Restatement) G is nonplanar **if and only if** G has K_5 or $K_{3,3}$ as a minor.

March 20 (Day 22)

Graph Complements:

The *complement* of a graph G, denoted either G^c or $\overline{G}$, is a graph with the same vertex set but that has every edge NOT in G.

A graph G is *self-complementary* if G^c is isomorphic to G. Examples: P_3 and C_5.

March 29 (Day 26)

de Bruijn Sequences:

The sequence 0000110101111001 is a sequence of length 16 that contains each of the sixteen binary sequences of length 4 (cycling allowed).

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

This is an example of a binary de Bruijn sequence. It is the most compact way we could represent these sixteen sequences.

Definition: A *sequence* is a succession of numbers $s_1 s_2 s_3 \dots s_l$.

Definition: The value l is called the *length* of the sequence.

Definition: A *binary sequence* is a succession of 0's and 1's.

Definition: A *de Bruijn* sequence of order n on the alphabet $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ is a sequence $S = s_1 s_2 \dots s_{k^n}$ of length k^n such that every sequence $b_1 b_2 \dots b_n$ of length n on $\mathcal{A}$ is a consecutive subsequence of S. That is, there exists an i with $1 \leq i \leq k^n$ such that $b_1 b_2 \dots b_n = s_i s_{i+1} s_{i+2} \dots s_{i+n-1}$.

Definition: A *binary de Bruijn sequence* of order n is a de Bruijn sequence of order n on the alphabet $\mathcal{A} = \{0, 1\}$.

Theorem: A de Bruijn sequence of order n on $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ always exists.

Proof: (using graph theory!)

Definition: A *de Bruijn graph* of order n on $\mathcal{A}$ is a directed *pseudo*graph that has as its nodes sequences of letters of length $n-1$. Each node has k out-edges, represented by the letters of the alphabet $\mathcal{A}$. Following edge a_i adds the letter a_i to the end of the sequence and removes the first letter from the sequence.

For example, $b_1 b_2 \dots b_{n-1} \xrightarrow{a_i} b_2 \dots b_{n-1} a_i$ and $b_1 b_2 \dots b_{n-1} \xrightarrow{a_j} b_2 \dots b_{n-1} a_j$.

You know you're done placing edges when ever vertex has k out-edges.

Proof: (continued) A de Bruijn sequence $\iff$ an Eulerian tour of the corresponding de Bruijn graph.

This is because each edge represents a unique n-letter sequence: the $n-1$ letters from the initial node of the edge plus the nth letter along the edge.

This graph has an Eulerian tour because the in-degree = out-degree of each vertex (the analogous result to the "Each vertex has even degree" theorem from Chapter 3.)

Ex. 1111011001010000 is a binary de Bruijn sequence of order 4.

Fact: There are $2^{2^{n-1}}$ binary de Bruijn sequences of order n.

Proof: Surprisingly, using determinants of Laplacians (those matrices from Day 26)!

Knight's Tours:

Definition: A *knight* refers to a chess piece moves two squares vertically and one square horizontally, or vice versa. Such a move is called a *knight move*.

Definition: A *(closed) knight's tour* is a succession of knight moves that visits each square on the chessboard exactly once (and returns to the first square).

Note: If you create a graph by drawing an edge between every two squares in the chessboard that are a knight move away, the problem of finding a knight's tour reduces to a problem of finding a Hamiltonian cycle in this graph. As we know, finding a Hamiltonian cycle in a graph is hard, but we do know on which $m \times n$ chessboards there is a knight's tour.

Theorem: If you have an $m \times n$ chessboard, where $m \leq n$, then there is a knight's tour unless one of the following holds.

1. m and n are both odd.
2. m equals 1, 2, or 4.
3. m equals 3 and n equals 4, 6, or 8.