Connectedness Definitions and Theorems from Week 6 (2/27–3/3)

February 27 (Day 16)

Connectedness Definitions:

Definition: A graph G is connected if for all $v, w \in V(G)$, there exists a path from v to w.
Definition: G is disconnected if G is not connected.
Definition: A (connected) component H is a MAXIMAL subgraph H of G that is connected.
Definition: G is k-connected if removing fewer than k vertices does not disconnect the graph.
Definition: The connectivity of G (denoted $\kappa(G)$ = "kappa") is the maximum k such that G is k-connected.

(Caveat: The connectivity of a single point is zero and $\kappa(K_n) = n - 1$.)

Definition: A cut vertex is a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected.
Definition: A SEPARATING SET or vertex cut or cut set is a set of vertices $X \subset V(G)$ such that $G \setminus X$ is disconnected.
Note: : $\kappa(G) = 0 \iff G$ is disconnected
Note: : $\kappa(G) \geq 2 \iff G$ has no cut vertex.

Edge Connectedness Definitions:

Definition: G is k-edge connected if removing fewer than k edges does not disconnect the graph.
Definition: The edge connectivity of G (denoted $\kappa'(G)$ = "kappa") is the maximum k such that G is k-edge connected.
Definition: A bridge is an edge $e \in E(G)$ such that $G \setminus e$ is disconnected.
Definition: A DISCONNECTING SET is a set of edges $D \subset E(G)$ such that $G \setminus D$ is disconnected.
Note: : $\kappa'(G) = 0 \iff G$ is disconnected.
Note: : $\kappa'(G) \geq 2 \iff G$ has no bridge.

Note: : If you remove a cut vertex, you increase the number of connected components.
Note: : If you remove a bridge, you increase the number of connected components by exactly one.

Theorems:

Theorem: (Book 2.4.1) Let G be connected. Then G is a tree $\iff$ Every edge of G is a bridge.
Theorem: (Book 3.2.1) A regular graph of even degree has no bridge.
Theorem: For all graphs G, $\kappa'(G) \leq \delta(G)$.

March 1 (Day 17)

Edge Cuts & A proof:

Note: $\kappa(G) \leq |V(G)| - 1$

Definition: Let $X \subset V(G)$. Then X^c is the complement of X.

That is, $V(G) = X \cup X^c$ and $X \cap X^c = \emptyset$.

Definition: For any $X \subset V(G)$ such that $X, X^c \neq \emptyset$, an edge cut (denoted $[X, X^c]$) is the set of edges D between X and X^c.

Note: An edge cut is a disconnecting set, but not vice versa.

Note: A minimal disconnecting set is an edge cut, but not vice versa.

Theorem: For all graphs G, $\kappa(G) \leq \kappa'(G)$.

Vertex Covers:

Definition: A vertex cover of a graph G is a subset $X \subset V(G)$ such that X contains (at least) one endpoint of every edge in G.

Definition: The size of the smallest vertex cover is denoted $\beta(G)$.

Theorem: In any graph G, $X \subset V(G)$ is an independent set $\iff$ X^c is a vertex cover.

Theorem: For all graphs G, $\alpha(G) + \beta(G) = |V(G)|$.

Discussion Section:

In discussion class, March 2, we remarked that the following theorem is true.

Theorem: If G has a Hamiltonian cycle, then $\kappa(G) \geq 2$ and $\kappa'(G) \geq 2$.

March 3 (Day 18)

Blocks:

Definition: A <u>block</u> of a graph G is a maximally connected subgraph of G with no cut vertex.
Note: The following things are true about blocks.

1. G itself may be a block.
2. Except for blocks that are edges, blocks are always 2-connected.
3. Any two blocks share at most one vertex.
4. A vertex shared between blocks is a cut vertex of G.
5. The blocks of G partition $E(G)$.

Definition: The <u>block graph</u> of G is a bipartite graph H with vertices v_i representing cut vertices of G, and vertices b_j representing blocks of G, where $v_i b_j$ is an edge of H if vertex v_i is a vertex in block b_j.
Note: A block graph is always a forest. (Proof in hwk.)

Characterization of 2-connectedness:

Theorem: (Whitney, 1932 aka MINI-Menger) Let G be a graph with ≥ 3 vertices. Then G is 2-connected $\iff$ for all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.
Theorem: (Menger) G is k-connected $\iff$ for all $v, w \in V(G)$, there exist k internally disjoint v, w-paths in G.
Definition: Let H be any subgraph of G. Then an <u>H-path</u> (or an <u>ear</u>) is a path in G that starts and ends in H.
Definition: An <u>ear decomposition</u> is a construction of G starting with some cycle C, and at each step successively adding to the existing graph H some H-path.
Theorem: G is 2-connected $\iff$ G has an ear decomposition.
Theorem: Let G be a graph with ≥ 3 vertices. The following are equivalent:

1. G is 2-connected.
2. G is connected and has no cut vertex.
3. G is a block.
4. For all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.
5. For all $v, w \in V(G)$, there exists a cycle in G through v and w.
6. $\delta(G) > 0$ and for all $e, f \in E(G)$, there exists a cycle in G through e and f.
7. G has an ear decomposition.