Math 356 - Dimensional Analysis Supplement

The following is a reformulation of two examples that appear in Chapter 8 done in terms of linear algebra. Comparing the work in each of these examples to the solutions provided in the text will prove to be enlightening in their own right. They are then followed by two abstract thinking questions that provide more tools to aid in effective model design.

Example 1. Let us study the propellent forces of a car by analyzing the dimensionless products of given variables F, the propellent force of the car, C, the consumption rate of gasoline, K, the amount of energy in a gallon of gas, and v, the velocity.

The product we are interested in studying is:

$$F^a C^b K^c v^d$$

Taking the MLT-decomposition of these quantities we obtain:

$$\begin{aligned} & (MLT^{-2})^a \ (L^3T^{-1})^b \ (ML^{-1}T^{-2})^c \ (LT^{-1})^d \\ = \ & M^{a+c} \ L^{a+3b-c+d} \ T^{-2a-b-2c-d} \end{aligned}$$

The dimensionless products occur precisely when the exponents on this expansion are precisely 0. That is, solve the following system of (homogeneous) equations:

$$\begin{aligned} a+c &= 0 \\ a+3b-c+d &= 0 \\ -2a-b-2c-d &= 0 \end{aligned}$$

We will accomplish this by calculating the Nullspace of the following:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 1 \\ -2 & -1 & -2 & -1 \end{bmatrix}$$

Row reducing $\mathbf{A}$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Giving us the following, simpler, linear system:

$$\begin{aligned} a-d &= 0 \\ b+d &= 0 \\ c+d &= 0 \end{aligned}$$

Also, the column corresponding to d has no leading 1, telling us that d is dependent in this system of equations. Let us assign d the value r giving us the following vector representation of this linear system:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Giving a 1-dimensional Nullspace. Any vector in $Span(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix})$ yield a dimensionless product. Since we are most interested in the case when $a = 1$ we will just take the vector we found. Using these a, b, c, and d values we see that

$$\Pi_1 = F^1 C^{-1} K^{-1} v^1 = M^0 L^0 T^0$$

By Buckingham's Theorem we know that there exists a function f such that $f(\Pi_1) = 0$, thus Π_1 is a root of this function, or, Π_1 is a constant. That tells us that

$$\begin{aligned} F^1 C^{-1} K^{-1} v^1 &= k \\ F &= k\,\frac{C\ K}{v} \\ F &\propto \frac{C\ K}{v} \end{aligned}$$

The same result from Chapter 2.

Example 2. Now let us consider a case where the dimension of the Nullspace involved is 2. Specifically, let's characterize the dimensionless products of a pendulum whose variables we will consider are length r, mass m, period t, gravity g, and angle of elevation θ. The product we are looking to analyze is:

$$m^a g^b t^c r^d \theta^e$$

The MLT-decomposition of this is:

$$\begin{aligned} & (M)^a\ (LT^{-2})^b\ (T)^c\ (L)^d\ (M^0 L^0 T^0)^e \\ = \ & M^{a+0e}\ L^{b+d+0e}\ T^{-2b+c+0e} \end{aligned}$$

Yielding the following system of equations:

$$\begin{aligned} a + 0e &= 0 \\ b + d + 0e &= 0 \\ -2b + c + 0e &= 0 \end{aligned}$$

The coefficient matrix of this and its row reduction is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

Thus the d and e correspond to the dependent columns. Letting $d = r$ and $e = s$ we get the following set of equations and its vector representation.

$$\begin{array}{rcl} a & = & 0 \\ b & = & -d \\ c & = & -2d \\ d & = & r \\ e & = & s \end{array} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = r \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This 2-dimensional Nullspace tells us that anything in the Span of these two vectors will yield a dimensionless product. We will use the vectors corresponding to $(r, s) = (0, 1)$ and $(r, s) = (-\frac{1}{2}, 0)$. Giving us the vectors $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ to form our dimensionless products

$$\begin{array}{rcl} \Pi_1 & = & g^{\frac{1}{2}} t^1 r^{-\frac{1}{2}} \\ \Pi_2 & = & \theta \end{array}$$

By Buckingham's Theorem we know that there exists a function f such that $f(\Pi_1, \Pi_2) = 0$. Since we are interested in understanding the period t let us suppose we can solve f for Π_1 (since t is in Π_1). That is, $\Pi_1 = h(\Pi_2)$ for some function h. Replacing the products we get:

$$\begin{array}{rcl} g^{\frac{1}{2}} t^1 r^{-\frac{1}{2}} & = & h(\theta) \\ t & = & h(\theta)\ g^{-\frac{1}{2}} r^{\frac{1}{2}} \\ t & = & \sqrt{\frac{r}{g}}\ h(\theta) \quad . \end{array}$$

Exercises:

1. a. In Example 2 discuss the complications of function $h(\theta)$ that appears in the final result. (The actual physical model is $t = 2\pi\sqrt{\frac{r}{g}}$)

1. b. Discuss the benefit of being able to choose any element(s) from the Nullspace to yield a dimensionless product. Specifically relate this to choosing $(r, s) = (-1, 0)$ (instead of $(-\frac{1}{2}, 0)$) in Example 2 above.

1. c. In Example 2 we make the assumption that we can solve for one of the dimensionless products appearing in Buckingham's Theorem. Expand your discussion in part a to analyze how you can deal with "complicated" vectors (very few 0's) resulting in a Nullspace basis calculation.

2. Let us introduce another dimension to our MLT system, call this dimension A. Let us consider the following variables in our model and their $MLTA$ decompositions given:

$$\begin{aligned} V &= ML^2A^{-1}T^{-3} \\ F &= MLT^{-2} \\ r &= L \\ C &= AT \end{aligned}$$

Find a basis for the Nullspace involved in solving the corresponding homogeneous linear system. Use Buckingham's Theorem to analyze the dimensionless product(s) equal to $V^0F^0r^0C^0$. (This is an actual physical relationship involving electric potential V (possibly in volts). Where C represents electric charge, and A the electric current measurement (ampere).)