§3.3 Combinatorial Interpretations

We learned that the number of ways to choose k objects out of n total objects is $\binom{n}{k}$. We can use this fact "backwards" by interpreting an occurrence of $\binom{n}{k}$ as the number of ways to choose k objects out of n.

Using a *combinatorial interpretation* of a numerical quantity lets us prove identities in a new way!

Theorem 3.3.2: For $0 \leq r \leq n$, $\binom{n}{r} = \binom{n}{n-r}$.

Analytic Proof:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$$

Combinatorial Proof:

We will show that both sides of the equation count the same quantity. Therefore they must be equal.

Q: In how many ways can we adopt r of n cats available for adoption at the shelter?

A:

Combinatorial Proofs

Equation (5.2): $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Analytic Proof:

Combinatorial Proof: In how many ways can we choose from n club members a committee of k members with a chairperson?

Because the two quantities count the same set of objects in two different ways, they are equal. $\square$

Combinatorial Proofs

Theorem 5.1.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Analytic Proof:

Combinatorial Proof: In how many ways can we choose k flavors of ice cream if n different choices are available?

Because the two quantities count the same set of objects in two different ways, they are equal. $\square$

Combinatorial Proofs

Theorem 3.3.3: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$

Analytic Proof: ???

Combinatorial Proof: Let us count the number of subsets of $\{1, 2, \ldots, n\}$ in two different ways:

Because the two quantities count the same set of objects in two different ways, they are equal. $\square$

—Worksheet—

Bijections

A useful method of proving that two sets A and B are equal is by way of a bijection, a rule or function that pairs up elements of A and elements of B.

Notation:
$f : A \to B$ "f is a function from A to B."
$f : a \mapsto b$ "f maps element $a \in A$ to element $b \in B$.

A bijection is more than a simple rule—

Definition: A function $f : A \to B$ is *one-to-one* or an *injection* if whenever $f(a) = f(b)$, then $a = b$.

Definition: A function $f : A \to B$ is *onto* or a *surjection* if for all $b \in B$, there is some $a \in A$ such that $f(a) = b$.

Definition: A *bijection* is a function that is *one-to-one* and *onto*.

The existence of a bijection proves that two sets are equinumerous.

Bijections

Simple Example: Prove that $\binom{n}{r} = \binom{n}{n-r}$.

Intuition: The LHS counts ways to choose r of n, while the RHS counts ways to choose $n-r$ from n. First, choose something concrete, such as choosing r or $n-r$ marbles out of a bag. Next, find a way to match up those two sets in a logical way.

Proof: Let A be the set of all ways to choose r out of a bag of n distinct marbles. Let B be the set of all ways to choose $n-r$ out of a bag of n distinct marbles.

We will show that the function f is a bijection, where $f : A \to B$ is defined by:

f is one-to-one:

f is onto:

This proves that f is a bijection.
Therefore $|A| = |B|$, as desired. $\square$

Bijections

Prove the following binomial identity:
$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$.

Bijective Proof: We consider the number of ways to choose a subset of $\{1, 2, \ldots, n-1, n\}$.

- The LHS counts subsets S with an even number of elements while the RHS counts odd subsets T.
- Construct a bijection f between $A := \{S\}$ and $B := \{T\}$: Let S be any subset with an even number of elements.
  - If $n \in S$, then define $f(S) = S \setminus \{n\}$.
  - If $n \notin S$, then define $f(S) = S \cup \{n\}$.
- Claim: This rule is a bijection.
  - Why is $f(S)$ in B?
  - Why is f one-to-one and onto?

For another proof, see pages 132–133.