Catalan Numbers

Four combinatorial objects to count:

1. Triangulations of an $n+2$-gon

2. Walks from $(0,0)$ to (n,n) staying above $y=x$

3. Sequences of length $2n$ with n $+1$'s and n -1's such that every partial sum is ≥ 0

4. Ways to multiply $n+1$ numbers together.

Catalan Numbers

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
1	1	2	5	14	42	132	429	1430	4862

These are four of the many many combinatorial interpretations of the Catalan numbers,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We claim that C_n is equal to:

triangulations $n+2$-gon	lattice paths $(0,0) \to (n,n)$	$+/-$ seq's w/pos. part. sums, len. n	mult. schemes $n+1$ numbers

So there should be bijections between the sets!

Bijection 1: triangulations $n+2$-gon $\longleftrightarrow$ multiplication schemes $n+1$ numbers

Rule: label all but one side of the $n+2$-gon. Start on the outside and work in. When you know two sides of a triangle, multiply them together. Determine the multiplication scheme of the last edge.

Catalan Numbers

Bijection 2: | mult. schemes $n+1$ numbers | $\longleftrightarrow$ | $+/-$ sequences of length n with positive partial sums |

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Bijection 3: | $+/-$ seq's, len. n pos. partial sums | $\longleftrightarrow$ | lattice path walks from $(0,0) \to (n,n)$ above $y=x$ |

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Number Formula

Goal: Prove the formula for Catalan numbers.

Proof 1. [via the triangulation interpretation of the Catalan numbers and recurrences]
Define $h_n = C_{n-1}$ to be the number of triangulations of an $n+1$-gon. We count this differently:

The $n+1$-st side must be involved in some triangle, defined by the third vertex of the triangle.
There are $n-1$ choices for this vertex. For each, determine in how many ways we can finish the triangulation:

Case v_i.
To the left:____-gon.
To the right:____-gon.

Hence, the total number of ways to triangulate is:

$$h_n = \sum_{i=} \qquad\qquad h_1 =$$

For example, $h_2 =$ $\qquad h_3 =$
$h_4 =$
What do these recurrences remind you of?

Catalan Number Formula

Suppose $g(x) = h_1 x + h_2 x^2 + h_3 x^3 + \cdots$
Then $g(x)^2 = (h_1 h_1)x^2 + (h_1 h_2 + h_2 h_1)x^3 + \cdots$.

$$g(x)^2 = \sum_{n \geq 0} \left(\sum_{i=1}^{n-1} h_i h_{n-i} \right) x^n$$

So, $g(x) - \Big(g(x)\Big)^2 =$

And we can solve this equation for $g(x)$.

Back on page 41 of the notes, we proved

$$\sqrt{1-4x} = 1 - 2 \sum_{n \geq 1} \tfrac{1}{n} \tbinom{2n-2}{n-1} x^n.$$

Therefore,

Catalan Number Formula

Proof 2. [a direct combinatorial approach using lattice paths] Define $C(x) = \sum_{n \geq 0} C_n x^n$. Weight each of the lattice paths by placing a weight of x on each North step and a weight of 1 on each East step. Therefore, each lattice path from $(0, 0)$ to (n, n) has weight ____. What is the weighted sum over all lattice paths of the weights of the lattice paths?______.

Let us count this quantity in another way. Either your path has a North step or it does not. The path with no North step has weight 1. Every other path starts with a North step and eventually returns to the line $y = x$ somewhere.

Schematic:

Therefore, $C(x) = 1 + xC(x)^2$.
Solving for $C(x)$, we have

[Explains all the comb. interp's of the Catalan #s!]