

Catalan Numbers

Four combinatorial objects to count:

1. Triangulations of an $n + 2$ -gon
2. Walks from $(0, 0)$ to (n, n) staying above $y = x$
3. Sequences of length $2n$ with $n + 1$'s and $n - 1$'s such that every partial sum is ≥ 0
4. Ways to multiply $n + 1$ numbers together.

Catalan Numbers

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
1	1	2	5	14	42	132	429	1430	4862

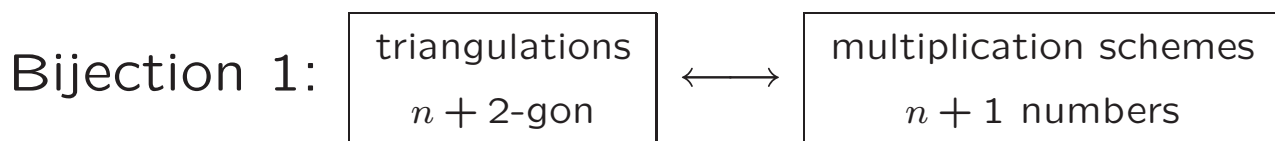
These are four of the many many combinatorial interpretations of the Catalan numbers,

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

We claim that C_n is equal to:

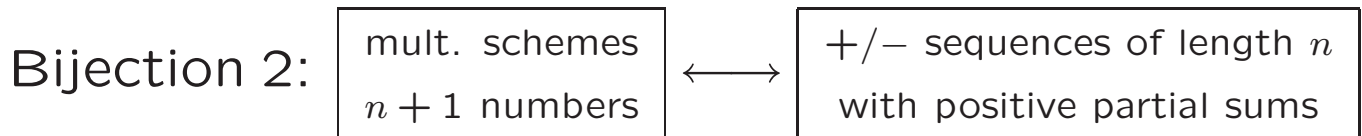
triangulations $n + 2$ -gon	lattice paths $(0, 0) \rightarrow (n, n)$	+/- seq's w/pos. part. sums, len. n	mult. schemes $n + 1$ numbers
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So there should be bijections between the sets!

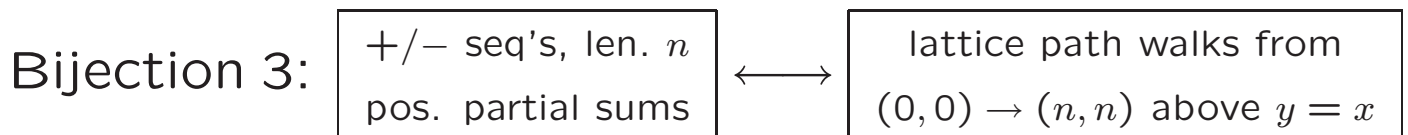


Rule: label all but one side of the $n + 2$ -gon. Start on the outside and work in. When you know two sides of a triangle, multiply them together. Determine the multiplication scheme of the last edge.

Catalan Numbers



Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1 's.



A sequence of $+$'s and $-$'s converts to a sequence of N 's and E 's, which is a path in the lattice.

Catalan Number Formula

Goal: Prove the formula for Catalan numbers.

Proof 1. [via the triangulation interpretation of the Catalan numbers and recurrences]

Define $h_n = C_{n-1}$ to be the number of triangulations of an $n + 1$ -gon. We count this differently:

The $n + 1$ -st side must be involved in some triangle, defined by the third vertex of the triangle.

There are $n - 1$ choices for this vertex. For each, determine in how many ways we can finish the triangulation:

Case v_i .

To the left: _____-gon.

To the right: _____-gon.

Hence, the total number of ways to triangulate is:

$$h_n = \sum_{i=1}^{n-1} h_i h_{n-i} \quad h_1 = 1$$

For example, $h_2 = 1$ $h_3 = 2$

$h_4 = 5$

What do these recurrences remind you of?

Catalan Number Formula

Suppose $g(x) = h_1x + h_2x^2 + h_3x^3 + \dots$

Then $g(x)^2 = (h_1h_1)x^2 + (h_1h_2 + h_2h_1)x^3 + \dots$

$$g(x)^2 = \sum_{n \geq 0} \left(\sum_{i=1}^{n-1} h_i h_{n-i} \right) x^n$$

So, $g(x) - (g(x))^2 =$

And we can solve this equation for $g(x)$.

Back on page 41 of the notes, we proved

$$\sqrt{1-4x} = 1 - 2 \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^n.$$

Therefore,

Catalan Number Formula

Proof 2. [a direct combinatorial approach using lattice paths] Define $C(x) = \sum_{n \geq 0} C_n x^n$. Weight each of the lattice paths by placing a weight of x on each North step and a weight of 1 on each East step. Therefore, each lattice path from $(0, 0)$ to (n, n) has weight _____. What is the weighted sum over all lattice paths of the weights of the lattice paths?_____.

Let us count this quantity in another way. Either your path has a North step or it does not. The path with no North step has weight 1. Every other path starts with a North step and eventually returns to the line $y = x$ somewhere.

Schematic:

Therefore, $C(x) = 1 + xC(x)^2$.
Solving for $C(x)$, we have

[Explains all the comb. interp's of the Catalan #'s!]