Partitions

A *partition* of a set S is a decomposition of S into disjoint non-empty parts (subsets)

$$S = A_1 \cup A_2 \cup \cdots \cup A_k; \quad A_i \neq \emptyset$$

A *partition* of an integer n is a decomposition of n as a sum of non-zero parts (integers)

$$n = n_1 + n_2 + \cdots + n_k; \quad n_i > 0$$

How this fits into our earlier framework:

	Placing p **distinguishable** objects	Placing p **indistinguishable** objects
into k **distinguishable** boxes		
into k **indistinguishable** boxes		

Counting Set Partitions

The *Stirling number of the second kind* counts the number of ways to partition a set of p elements into k non-empty subsets. Notation: $S(p,k)$ or $\left\{ {p \atop k} \right\}$.

$p \backslash k$	$\left\{ {p \atop 0} \right\}$	$\left\{ {p \atop 1} \right\}$	$\left\{ {p \atop 2} \right\}$	$\left\{ {p \atop 3} \right\}$	$\left\{ {p \atop 4} \right\}$	$\left\{ {p \atop 5} \right\}$	$\left\{ {p \atop 6} \right\}$
0	1						
1		1					
2		1	1				
3		1	3	1			
4		1	7	6	1		
5		1	15	25	10	1	
6		1	31	90	65	15	1

Do you notice any patterns?

Recurrence: $\left\{ {p \atop k} \right\} =$

In how many ways can we place p objects into k boxes? Condition on the placement of element #1:

Stirling number formula

Goal: Find a <u>formula</u> for $S(p,k)$.
We know $S(p,k) =$ number of ways to put p distinguishable objects in k non-empty **indistinguishable** boxes.
Define $S^{\#}(p,k) =$ number of ways to put p distinguishable objects in k non-empty **distinguishable** boxes.
Then $S(p,k)$ and $S^{\#}(p,k)$ are related by:

Method: Use Inc/Exc to find a formula for $S^{\#}(p,k)$:
(condition on whether Box 1 ... Box k is empty.)

For all i, let A_i be the set of all ways to place p objects in k boxes if Box i is empty.

Then $S^{\#}(p,k) = |S| - |A_1 \cup A_2 \cup \cdots \cup A_k|$;
$|S| =$
$|A_i| =$
$|A_i \cap A_j| =$
Therefore,

$$\begin{aligned} S^{\#}(p,k) &= \_\_ - \_\_\ \_\_\_\_ + \_\_\ \_\_\_\_ - \cdots \\ &= \sum_{t=0}^{k} (-1)^t \binom{k}{t} (k-t)^p \end{aligned}$$

So, $S(p,k) = \dfrac{1}{k!} \displaystyle\sum_{t=0}^{k} (-1)^t \binom{k}{t} (k-t)^p$

Bell Numbers

Let B_p be the number of partitions of a set with p elements. (Any number of non-empty parts.)

$$B_p = \left\{ {p \atop 0} \right\} + \left\{ {p \atop 1} \right\} + \left\{ {p \atop 2} \right\} + \cdots + \left\{ {p \atop p} \right\}$$

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
1	1	2	5	15	52	203	877	4140	21147

The Bell numbers satisfy a recurrence: (Thm 8.2.7)

$$B_p = \binom{p-1}{0} B_0 + \binom{p-1}{1} B_1 + \cdots + \binom{p-1}{p-1} B_{p-1}$$

We prove this combinatorially by conditioning on the box containing the last element p:

How many partitions of $\{1, \cdots, p\}$ are there if there are k elements in the box with p?

Partitions of Integers

A partition of an integer n is a decomposition of n indistinguishable objects into indistinguishable boxes. 3+1+1, 1+3+1, and 1+1+3 are all the same, so we will write the numbers in non-increasing order.

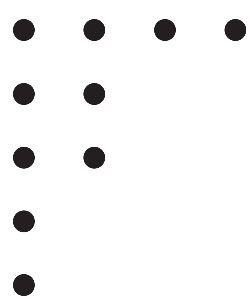
When we discuss partitions of an integer, (say n) we often use the letters λ, μ, and ν. We'll write:

$$\lambda : n = n_1 + n_2 + \cdots + n_k \text{ or } \lambda \vdash n.$$

For example,
$\lambda : 5 = 3{+}1{+}1,$ or $\lambda = 311,$ or $\lambda = 3^1 1^2,$ or $311 \vdash 5.$

A pictoral representation of $\lambda = n_1 n_2 \cdots n_k$ is its *Ferrers diagram*, a left-justified array of dots with k rows, containing n_i dots in row i.

Example: $42211 \vdash 10$
is represented by



The *conjugate* of a partition λ is the partition λ^* which interchanges rows and columns.
Some partitions are *self-conjugate*, satisfying $\lambda = \lambda^*$.

Counting Partitions

Let p_n denote the number of partitions of n. Then,

p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
1	1	2	3	5	7	11	15	22	30

These numbers have a very ugly formula but a very beautiful generating function.

$$P(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\, \frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right],$$

Think about breaking down the integer n into some number of parts of size 1, size 2, size 3, etc. We can write down a generating function for each size and then multiply them together.

Parts of size 1:

Parts of size 2:

Parts of size k:

Therefore, $p(x) = \sum_{n=0}^{\infty} p_n x^n =$

Theorems involving Partitions

Theorem: The number of partitions q_n of n with no parts equal to 1 is $p_n - p_{n-1}$.

Proof. The generating function for $\{q_n\}$ is:

$$
\begin{aligned}
q(x) &= \frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\cdots \\
&= (1-x)p(x) \\
&= (1-x)\left(\sum_{n\geq 0} p_n x^n\right) \\
&= \left(\sum_{n\geq 0} p_n x^n\right) - \left(\sum_{n\geq 0} p_n x^{n+1}\right) \\
&= \left(\sum_{n\geq 0} p_n x^n\right) - \left(\sum p \qquad x \qquad\right) \\
&=
\end{aligned}
$$

Theorem: The number of partitions of n whose largest part is k is equal to the number of partitions of n with k parts.

Proof.

Theorems involving Partitions

Theorem: The number of partitions d_n of n into distinct, odd parts, is equal to the number of self-conjugate partitions of n.

Investigation: Does this make sense? For $n = 6$, d_6:

Proof 1: (Gen. func.)

Proof 2: (Bijective)

Standard Young Tableaux

The following material is very near to some current lines of research in algebra and combinatorics.

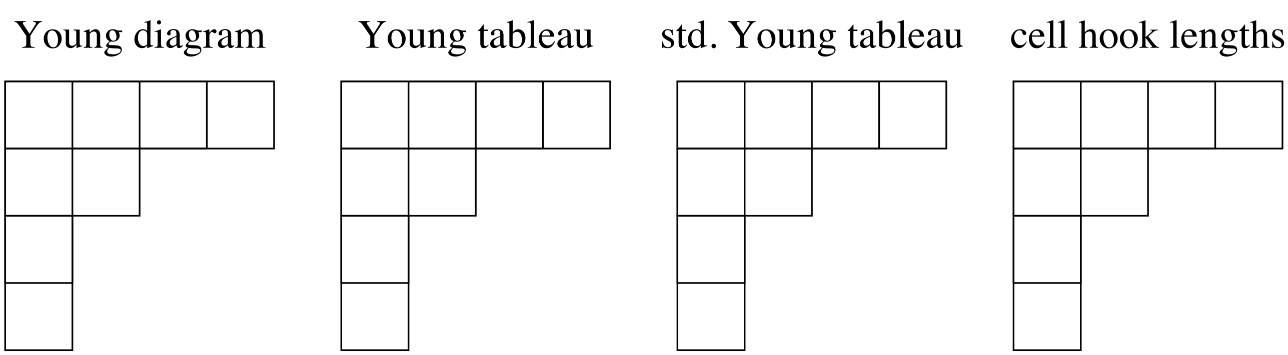
Definitions:

A *Young diagram* is a representation of a partition using left-justified boxes called cells.

A *Young tableau* is a placement of positive integers into the boxes of a Young diagram.

A *standard Young tableau* is a Young tableau where the integers are 1 through n, and both the rows and the columns are increasing.

The *hook length* $h(i,j)$ of a cell (i,j) is the number of cells in the "hook" to the left and down.



Q:How many SYT are there of shape $\lambda \vdash n$?

A: $\dfrac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$