Solving Recurrence Relations using Generating Functions

Example. Solve the recurrence relation
$h_n = 5h_{n-1} - 6h_{n-2}$ with i. c. $h_0 = 1$ and $h_1 = -2$.

Define $g(x) = \sum_{n\geq 0} h_n x^n$. Then we know

$$
\begin{aligned}
g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_n x^n + \cdots, \\
-5xg(x) &= \\
6x^2 g(x) &=
\end{aligned}
$$

Therefore, $g(x) =$

We can determine a formula for h_n using partial fractions.

$$\frac{1-7x}{(1-3x)(1-2x)} = \frac{A}{1-3x} + \frac{B}{1-2x}$$

Because
$1/(1-3x) = 1 + (3x) + (3x)^2 + \cdots = \sum_{n\geq 0} 3^n x^n$ and
$1/(1-2x) = 1 + (2x) + (2x)^2 + \cdots = \sum_{n\geq 0} 2^n x^n$,
We know $g(x) = \sum_{n\geq 0} \Big(\quad 3^n + \quad 2^n \Big) x^n$

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With repeated roots, the method of partial fractions still works but its result is not quite as nice.

Example: Find the partial fraction decomposition of $x/(1-2x)^2(1+5x)$.

Since $(1-2x)^2$ is a repeated root,

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{(1+5x)}.$$

Clearing the denominator gives:

$$x = A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2.$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 0 + B(1+\frac{5}{2}) + 0$; so $B = \frac{1}{7}$.
When $x = -\frac{1}{5}$, $-\frac{1}{5} = 0 + 0 + C(1+\frac{2}{5})^2$; so $C = \frac{-5}{49}$.

Equating the coefficients of 1, we know that $A + B + C = 0$, so we can conclude $A = \frac{-2}{49}$.

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{-\frac{2}{49}}{(1-2x)} + \frac{\frac{7}{49}}{(1-2x)^2} + \frac{-\frac{5}{49}}{(1+5x)}.$$

Solving Recurrence Relations

Example: Let $\{h_n\}_{n\geq 0}$ be a sequence satisfying

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0,$$

with initial conditions $h_0 = 1$, $h_1 = 1$, and $h_2 = -1$. Find a generating function and formula for h_n.

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_n x^n + \cdots, \\ +xg(x) &= \\ -16x^2 g(x) &= \\ +20x^3 g(x) &= \end{aligned}$$

Therefore, $g(x) =$

We recall that $(1-y)^{-m} = \sum_{n\geq 0} \binom{m+n-1}{n} y^n$. Therefore, $\frac{1}{(1-2x)^2} = \sum_{n\geq 0} \binom{n+1}{n} (2x)^n = \sum_{n\geq 0} (n+1) 2^n x^n$

and we conclude that

$h_n =$