A Taste of Generating Functions

Example: Find the sequence $\{a_n\}_{n\geq 0}$ that satisfies $a_0 = 0$ and the recurrence $a_{n+1} = 2a_n + 1$ for $n \geq 0$.

We will use generating functions to determine a formula for a_n. Define $A(x) = \sum_{n\geq 0} a_n x^n$.

Step 1: Multiply both sides of the recurrence by x^n and sum over all n:

$$\sum_{n\geq 0} a_{n+1} x^n = \sum_{n\geq 0} (2a_n + 1) x^n$$

Step 2: Massage the sums to find copies of $A(x)$.

On the LHS: need power of x to equal term index;
On the RHS: separate into pieces with, w/o a_n.

$$\frac{1}{x} \sum_{n\geq 0} a_{n+1} x^{n+1} = \sum_{n\geq 0} 2a_n x^n + \sum_{n\geq 0} x^n$$

Therefore,

Important G.F. Manipulations

Key series:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \qquad \frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{n} x^n$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!} \qquad (1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k$$

Manipulations on $A(x) = \sum_{n\geq 0} a_n x^n$:

$$[x^n]\Big(x^b A(x)\Big) = [x^{n+b}]\Big(A(x)\Big)$$

$$\sum_{n\geq 1} a_{n-1} x^n =$$

$$\sum_{n\geq 0} a_{n+1} x^n =$$

$$\sum_{n\geq 0} n a_n x^n =$$

$$\sum_{n\geq 0} p(n) a_n x^n = p\Big(x\frac{d}{dx}\Big)\Big(A(x)\Big)$$

Example: Find $\displaystyle\sum_{n\geq 0} \frac{n^2+4n+5}{n!}$

Additional G.F. Manipulations

Let $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$.

Then what is the coefficient of x^n of $A(x)B(x)$?

Upon expanding the product, multiplying a term $a_k x^k$ in A and a term $b_l x^l$ in B only contribute to the coefficient of x^n if ______________________.

Therefore, $A(x)B(x) = \sum_{n \geq 0}$

One interpretation of this theorem:
If a_n counts all sets of size n of type "S",
and b_n counts all sets of size n of type "T",
then $[x^n]\Big(A(x)B(x)\Big)$ counts all pairs of sets (S, T) where the total number of elements in both sets is n.

A special case: $(A(x))^2 = \sum_{n \geq 0}$

Similar to above, the coefficient of x^n of $(A(x))^k$ is: $[x^n]\Big((A(x))^k\Big) = \sum_{n_1+n_2+\cdots+n_k=n} a_{n_1} a_{n_2} \cdots a_{n_k}$

Rolling Dice

Example: (a) When two standard six-sided dice are rolled, what is the distribution of the sums?
(b) Is it possible to create two six-sided dice with non-standard labelings such that when we roll them we recover the same distribution of sums?

Define $D(x)$ to be a generating function that represents the roll of one die. In other words,
$[x^n]\Big(D(x)\Big) =$ the number of ways in which n arises.
So, $D(x) =$ Notice $D(1) = 6$

Q: What does $[x^n]\Big(D(x)^2\Big)$ represent?
A: The number of ways that a total of n dots can appear on die 1 and die 2. So the answer to part (a) is $D(x)^2 =$

Part (b): The distribution of the sums is $D(x)^2$, so this question asks us to:

Fruit Baskets

Example: Use a generating function method to determine in how many ways we can create a fruit basket with n pieces of fruit, where we have an infinite supply of apples and bananas, with the added constraints:

- The number of apples is even.
- The number of bananas is a multiple of five.
- The number of oranges is at most four.
- The number of pears is zero or one.

STRATEGY: Write down a power series for each piece of fruit to represent the possible numbers of fruit we can put into the basket.