Integer Sequences

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

Integer sequences often arise as a compilation of experimental combinatorial data. We will develop tools to allow us to find a formula for the n^{th} *term*.

$$a_n = f(n)$$

In general, this is a hard. Let's start off easy:

arithmetic: $a_0, a_0 + q, a_0 + 2q, a_0 + 3q, \ldots$
geometric: $a_0, \quad qa_0 \quad , \quad q^2 a_0 \quad , \quad q^3 a_0 \quad , \ldots$

Example: Let h_n be the number of regions created by n mutually overlapping circles.
We can calculate the first few terms.

We can make a conjecture for the formula.

(Or a relation that successive terms satisfy.)

Integer Sequences

These methods may be considered "experimental" in that we are collecting data. However, the underlying answers will always be exact.

A sequence $\{h_n\}_{n \geq 0}$ satisfies a *recurrence relation of order* k if there are quantities b_n and $a_1, \ldots, a_k$ with $a_k \neq 0$ (that may depend on n) such that

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n \text{ (for } n \geq k)$$

A sequence is completely defined by a recurrence relation of order k along with k initial conditions.

Example: Starbucks is moving into a new town. The first month is spent hiring and training people. Starting the second month and then every month thereafter, a manager is created that opens a new Starbucks the following month. Determine how many branches there will be after n months.

§7.1 Fibonacci Numbers

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}
1	1	2	3	5	8	13	21	34	55	89	144	233

One tool: Online Encyclopedia of Integer Sequences
`http://www.research.att.com/~njas/sequences/`

The Fibonacci numbers satisfy $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 1$ and $f_1 = 1$.
Note: This is NOT a formula!

Identities involving a sequence satisfying a recurrence are most easily proven using induction.
Example: f_n is even iff n is divisible by 3.

There is a combinatorial interpretation of the Fibonacci numbers in terms of square-domino tilings:

Let b_n be the number of ways to tile an $n \times 1$ board using squares and dominoes. Calculate a few terms:

We can guess:

§7.1 Fibonacci Numbers

Two sequences are equal if they satisfy the same recurrence and have the same initial conditions. We see that $b_0 =$ and $b_1 =$

Why is $b_n = b_{n-1} + b_{n-2}$?

We can use these "offset" Fibonacci numbers to prove Fibonacci identities combinatorially.

Theorem: $b_{2n} = b_n^2 + b_{n-1}^2$ $\left[\text{Or: } f_{2n+1} = f_{n+1}^2 + f_n^2.\right]$

For more combinatorial proofs involving the Fibonacci sequence and related numbers, check out: *Proofs that Really Count: The Art of Combinatorial Proof*, by Benjamin and Quinn.

Generating Functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

— *Generatingfunctionology*, H. S. Wilf

For any sequence $\{a_n\}_{n\geq 0} = a_0, a_1, a_2, a_3, \ldots$, its *generating function* is the formal power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n\geq 0} a_n x^n.$$

Example: Let f_n be the Fibonacci numbers. Then

$$F(x) = \sum_{n\geq 0} f_n x^n = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots .$$

We will see that we can simplify this expression greatly. In fact, $F(x) = \dfrac{x}{1-x-x^2}$.

Conversely, we would say that f_n is the coefficient of x^n in $\frac{x}{1-x-x^2}$. We write this as $f_n = [x^n]\left(\frac{x}{1-x-x^2}\right)$.

Example: Let $A(x) = 1/(1+x)$. We recognize this as the geometric series $1/(1-(-x))$; therefore, $A(x) = 1 - x + x^2 - x^3 + \cdots$ and $[x^n]A(x) = (-1)^n$.

Generating Functions

We will use generating functions to:

- Find an exact formula for a sequence.
- Prove identities involving a sequence.
- Understand partitions of an integer.

Others use generating functions to:

- Understand the asymptotics of a sequence.
- Find averages and statistical properties.
- Understand *something* about a sequence.

—Worksheet—