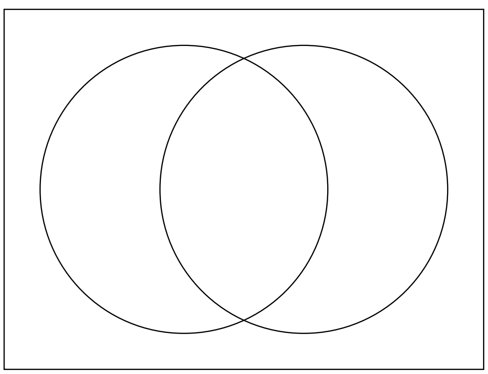
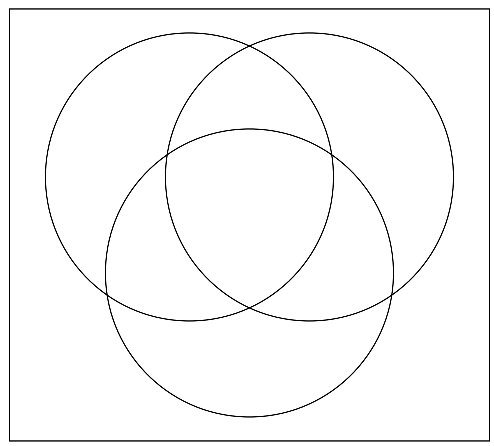
§6.1 Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k$ and the A_i are pairwise disjoint, we know that $|A| = |A_1| + \cdots + |A_k|$.

When $A = A_1 \cup \cdots \cup A_k$ and the subsets A_i are NOT pairwise disjoint, we must apply the principle of *inclusion-exclusion* in order to determine $|A|$.

$k = 2$	$k = 3$
	
$\lvert A\rvert = \lvert A_1\rvert + \lvert A_2\rvert -$	$\lvert A\rvert = \lvert A_1\rvert + \lvert A_2\rvert + \lvert A_3\rvert$
$\lvert S - A\rvert =$	$\lvert S - A\rvert =$

§6.1 mmm...PIE

Theorem 6.1.2: $|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| - \cdots + (-1)^{m+1}\left|A_1 \cap \cdots \cap A_m\right|$

Theorem 6.1.1: $|\overline{A_1} \cup \cdots \cup \overline{A_m}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap \cdots \cap A_m|$

The key to using the principle of inclusion-exclusion is determining sets A_i that are easy to count.

Example: Find the number of integers between 1 and 1000 that are not divisible by 5, 6, or 8.

Solution: Let $S = \{n \in \mathbb{Z} \text{ s.t. } 1 \leq n \leq 1000\}$
Let $A_1 \subset S$ be the multiples of 5,
$A_2 \subset S$ be the multiples of 6, and
$A_3 \subset S$ be the multiples of 8.

Then, <u>in words</u>, $A_1 \cap A_2$ contains

$A_1 \cap A_3$ $A_2 \cap A_3$

and $A_1 \cap A_2 \cap A_3$ contains

$|A_1| =$ $|A_2| =$ $|A_3| =$

$|A_1 \cap A_2| =$ $|A_1 \cap A_3| =$

$|A_2 \cap A_3| =$ $|A_1 \cap A_2 \cap A_3| =$

So $|\overline{A_1} \cup \overline{A_2} \cup \overline{A_3}| =$

Example: Find how many permutations of "MATHISFUN" contains none of the words "MATH", "IS", or "FUN" as subwords.

§6.2 Combinations with Repetitions

Counting r-combinations of $\{1\cdot a_1, 1\cdot a_2, \cdots, 1\cdot a_k\}$
Counting r-combinations of $\{r\cdot a_1, r\cdot a_2, \cdots, r\cdot a_k\}$

Now we want to be able to count r-combinations of an arbitrary multiset. It's as easy as PIE.

Example: Determine the number of 10-combinations of the multiset $T = \{3\cdot a, 4\cdot b, 5\cdot c\}$.

Game plan: Let S be the set of 10-combinations of $\{\infty\cdot a, \infty\cdot b, \infty\cdot c\}$. Use inclusion/exclusion to find $|T| = |S - (A_1 \cup \cdots \cup A_k)|$ for the right sets A_i.

In choosing the A_i, we want 10-combinations that <u>violate</u> the conditions of T:
Define A_1 to be
A_2: A_3:
Then $A_1 \cap A_2$ are those 10-combs that
$A_1 \cap A_3$: $A_2 \cap A_3$: $A_1 \cap A_2 \cap A_3$
Therefore, $|A_1| =$
$|A_2| =$ $|A_3| =$ $|A_1 \cap A_2| =$
$|A_1 \cap A_3| =$ $|A_2 \cap A_3| =$ $|A_1 \cap A_2 \cap A_3| =$
$|\overline{A_1} \cup \overline{A_2} \cup \overline{A_3}| =$

Example: Find the number of integral solutions of $x_1 + x_2 + x_3 + x_4 = 16$ subject to $1 \le x_1 \le 5$, $-2 \le x_2 \le 4$, $0 \le x_3 \le 5$, and $3 \le x_4 \le 9$.

§6.3 Derangements

At a party, 10 gentlemen check their hats. They "have a good time", and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his own hat?

This is a *derangement* of ten objects.

Q: How many words (permutations) are there of $[n]$ such that the first letter is not 1, the second is not 2, and so on up to the last letter is not n?

A: Call this number D_n; we'll find a formula using inclusion/exclusion:
Let S be the set of all ways we can return the hats. Now let A_1 be the ways in which "1" gets his hat, let A_2 be the ways in which "2" gets his hat, etc. and let A_n be the ways in which "n" gets his hat. Then $D_n = |S| - |A_1 \cup \cdots \cup A_n|$.
$|S| =$ $|A_i| =$

When intersecting k sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$
The number of such intersections is

Therefore, $D_n =$

§6.3 More derangements

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \Big[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \Big] \end{aligned}$$

Taylor series expansion of e^x:

$e^x = 1 + \frac{x}{1!} + \frac{x}{2!} - \frac{x}{3!} + \cdots$. Therefore,

$$\approx \Big[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \Big]$$

Q: If the hats are passed back randomly, what is the probability that no one gets his hat back?

We can prove identities involving D_n combinatorially.

$$D_n = (n-1)\Big(D_{n-2} + D_{n-1} \Big)$$

§6.4 Permutations with Forbidden Positions

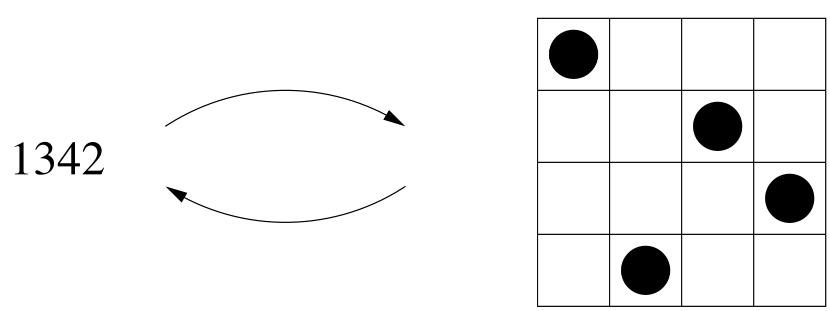
We can think of *derangements* as permutations of $[n]$ with the following restrictions:

in position 1, '1' can not appear.
in position 2, '2' can not appear.
in position n, 'n' can not appear.

A natural generalization is to find permutations of $[n]$ with the restrictions:

in position 1, none of $X_1 \subset [n]$ may appear.
in position 2, none of $X_2 \subset [n]$ may appear.
in position n, none of $X_n \subset [n]$ may appear.

We appeal to the framework of non-attacking rooks. For example, when $n = 4$, set up a 4×4 board:



There is a bijection between permutations of $[n]$ and placements of n non-attacking rooks on an $n \times n$ board.

§6.4 Forbidden Positions

perm's satisfying forbidden positions	$\leftrightarrow$	non-attacking rook placements w/ position restrictions

Example: Count and enumerate all permutations for $n = 4$ subject to the forbidden positions:

$$X_1 = \{1,2\},\ X_2 = \{2,3\},$$
$$X_3 = \{3,4\},\ X_4 = \{1,4\}.$$

×	×		
	×	×	
		×	×
×			×

×	×		
	×	×	
		×	×
×			×

×	×		
	×	×	
		×	×
×			×

§6.4 Forbidden Positions

For few forbidden positions, use Inclusion/Exclusion. Let I be the set of forbidden positions (squares).

For all $i \in I$, let A_i be the number of rook placements with one rook in forbidden position i.

Calculate $|A_i| =$

The ingenious part comes in when we have to calculate the size of the intersections:
In words, $A_i \cap A_j$ is

Therefore, $|A_i \cap A_j| =$

Similarly, $|A_i \cap A_j \cap A_k| =$

So, the number of permutations that respect the forbidden positions is
$|S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots =$
___ − ___ ____ + ___ ____ − ___ ____ + $\cdots$

Example: Find the number of permutations of [5] satisfying these forbidden positions:

	×			
×		×		
	×			
			×	×
			×	