Extensions of Binomial Coefficients

We already learned that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
A better way to calculate $\binom{n}{k}$ is as $\frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots \quad 1}$.

This allows us to define $\binom{r}{k}$ when $r \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots \quad 1}$$

and define: $\binom{r}{0} = 1$ $\qquad \binom{r}{k} = 0$ if $k < 0$

With this definition, for any $r \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$, $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ holds (as do most identities).

Proof of Pascal's Formula with r real:

When $r = -n \in \mathbb{Z}_{<0}$, $\qquad \binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

§5.6 Newton's binomial theorem

There is an $\mathbb{R}$-analogue for the binomial theorem:

For $\alpha \in \mathbb{R}$ and all x and y satisfying $0 \leq |x| < |y|$,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

We recover the binomial theorem as a special case:

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

An analog of $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for $\alpha \in \mathbb{R}$ is:

$$(z+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \text{ when } |z| < 1.$$

Example: Find the power series expansion of $\sqrt{1-4x}$.

$$\begin{aligned}
\sqrt{1-4x} &= \Big((-4x)+1\Big)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} (-4x)^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!} (-1)^k 4^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k! 2^k} (-1)^k 4^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k} \frac{(2k-2)!}{(k-1)!(k-1)!} x^k
\end{aligned}$$

Geometric Series

Let's find the sum of an infinite geometric series:

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{with } |x| < 1$$

Another way to see it: $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

$$\frac{1}{1-z} = (1 + (-z))^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} (-z)^k =$$

$$\sum_{k=0}^{\infty} (-1)^k \binom{k}{k} (-z)^k = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + \cdots$$

A preview of formal power series:

$$\frac{1}{(1-z)^n} = (1 + (-z))^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-z)^k =$$

$$\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} (-z)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k$$

Now in a Combinatorial Proof

Theorem: $\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k$

Proof: The left hand side is a product of n copies of $(1 + z + z^2 + z^3 + \cdots)$. We prove that the two infinite polynomial expressions are equal by showing that the coefficients of each monomial are equal.

So, show that the coefficient in of the expansion of the LHS equals $\binom{n+k-1}{k}$. In how many ways can we generate a term of the form z^k in the expression of the left hand side?

A term in the expansion of the left hand side is of the form

§5.5 Multinomials and the multinomial theorem

The binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
The multinomial theorem has the same form:

$$(x_1 + x_2 + \cdots + x_t)^n = \sum \left(\ ?? \ \right) x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

So we ask in how many ways can we create a term of the form $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$—that is, make a word with n_1 indistinguishable x_1's,
with n_2 indistinguishable x_2's, ... and so on,
and with n_t indistinguishable x_t's?

This is the permutation of the multiset
$\{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_t \cdot x_t\}$.

Therefore, define the *multinomial coefficient* to be

$$\binom{n}{n_1\, n_2\, \cdots\, n_t} = \frac{n!}{n_1! n_2! \cdots n_t!},$$

where $n_1 + n_2 + \cdots + n_t = n$.

Example: Find the coefficient of $x^5 y^2 z^3$ in $(2x + y - z)^{10}$.