§5.1 Pascal's Formula

We saw that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
This, with initial conditions allow us to calculate $\binom{n}{k}$ for all n and k in a recursive manner.
$\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all n.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1							1

Many fun sequences appear in Pascal's triangle:

Sequence		Name	ID
$1, 2, 3, 4, 5, \ldots$	$\binom{n}{1}$	$(a_n = n)$	A000027
$1, 3, 6, 10, 15, \ldots$	$\binom{n}{2}$	triangular	A000217
$1, 4, 10, 20, 35, \ldots$	$\binom{n}{3}$	tetrahedral	A000292
$1, 2, 6, 20, 70, \ldots$	$\binom{2n}{n}$	cent. binom.	A000984

`http://www.research.att.com/~njas/sequences/`

§5.1 Pascal's Formula and Induction

Pascal's formula is useful to prove identities by induction.

Example: $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ (*)

Proof: (by induction on n)

1. Base case: The identity holds when $n = 0$:

2. Inductive step: Assume that the identity holds for $n = k$ (inductive hypothesis) and prove that the identity holds for $n = k + 1$.

$\binom{k+1}{0} + \binom{k+1}{1} + \cdots + \binom{k+1}{k+1} =$

By induction, the identity holds for all $n \geq 0$.

§5.2 Binomial Coefficients

Theorem 5.2.1: (The binomial theorem.)
Let n be a positive integer. For all x and y,
$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$.

Let's rewrite in summation notation! Determine the generic term $[\binom{n}{k}x^{\quad}y^{\quad}]$ and the bounds on k

$$(x+y)^n = \sum$$

That is, the entries of Pascal's triangle are the coefficients of terms in the expansion of $(x+y)^n$.

A combinatorial proof of the binomial theorem:

Q: In the expansion of $(x+y)(x+y)\cdots(x+y)$, how many of the terms are $x^{n-k}y^k$?

A: You must choose y from exactly k of the n factors. Therefore, $\binom{n}{k}$ ways. $\square$

§5.2 Binomial Identities

Special cases of the binomial theorem:

- $y = 1$: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ (*)

- $x = y = 1$: $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

That is, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- $x = y = 1$: $(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$

That is, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} =$

Manipulating the binomial theorem generates (and proves) various fun binomial identities.

- Derivative, integral of (*) w.r.t. x: (& Plug in)

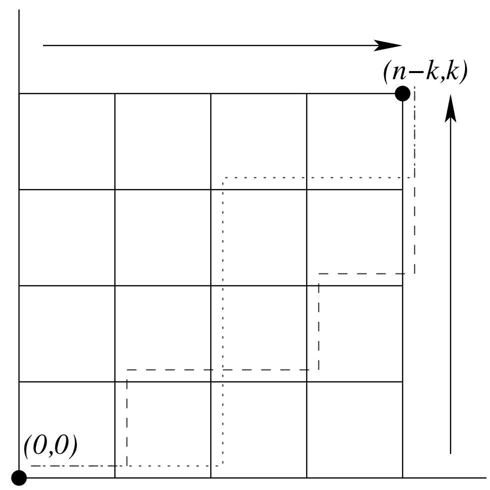
$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

$$\frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}$$

§5.1 Lattice Paths

We now can prove binomial identities
• analytically (factorials) • inductively (Pascal's)
• combinatorially (choose k of n objects)

Another combinatorial interpretation is in terms of paths in this square lattice:



Each path will start at the origin $(0, 0)$ and step either East [*a* $(0, 1)$ *step*] or North [*a* $(1, 0)$ *step*].

Q: How many different lattice paths are there from $(0, 0)$ to $(n - k, k)$? [Notation: $p(n, k)$.]

A1: $p(n, k)$ counts words with k E's and $n - k$ N's.

A2: $p(n, k) = p(n-1, k)+p(n-1, k-1)$ & $p(n, 0) = 0$