§3.4–3.5 Permutations and Combinations of Multisets

A *multiset* M is like a set where the members may repeat. Example:

$$M = \{a, a, a, b, c, c, d, d, d, d\}$$

– or we can write –

$$M = \{3 \cdot a, 1 \cdot b, 2 \cdot c, 4 \cdot d\}$$

Definition: *repetition numbers* of the members

We allow infinite repetition, in which case we would write $\infty \cdot a$.

We can ask

(and we will answer):

Q: How many r-permutations, permutations, and r-combinations are there of a certain multiset?

Remember the difference:

r-permutations order r elements of M.

permutations order all elements of M.

r-combinations choose r elements of M.

(A *multisubset*, in other words!)

§3.4–3.5 Permutations and Combinations of Multisets

Example: r-permutations when each element of M has an infinite repetition number:

Example: r-combinations when each element of M has an infinite repetition number:

Example: permutations of a finite multiset M:

§3.4 Permutations of a Finite Multiset

Let M have k different types of members with finite repetition numbers n_1 through n_k and let $|M| = n = n_1 + \cdots + n_k$.

How many permutations of S are there?

Proof 1: Label all the balls uniquely; how many permutations exist? ___ Now ignore the labelings. How many times does the same multiset permutation appear? [*What are the symmetries?*]

Proof 2: How many ways are there to place the balls of type 1? ____ Once they are placed, how many ways are there to place the balls of type 2?___

§3.4 Finite-Multiset-Permutation Examples

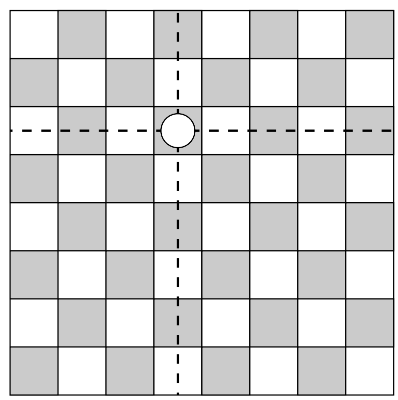
Example: How many permutations of the letters in Mississippi are there?

Example: In how many ways can we place n labeled objects into k labeled boxes, where each box B_i contains n_i objects and $n_1 + n_2 + \cdots + n_k = n$?

Example: What if all the boxes are all the same size and not labeled?

§3.4 Non-attacking Rooks

In chess, a *rook* is a piece that moves in a column or in a row. Two rooks are said to *attack each other* if they are in the same row or column.



Q: How many ways are there to place eight non-attacking rooks on an 8×8 chessboard?

A: There can not be two in the same row, so there must be one in each row. How many ways are there to place a rook in the first row? ______
Once placed, how many ways for the second? ___

In all?

§3.4 More Non-attacking Rooks

Q: In how many ways can eight distinguishable rooks be placed? (Eight different colors, perhaps.)

Q: What if there are four yellow rooks, three blue rooks, and one red rook?

Q: What if we wanted to place six indistinguishable rooks on an 8×8 chessboard?

§3.5 Infinite-Multiset-Combination Examples

The situation: we are looking for the number of r-combinations of a multiset

$$M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$

Our original interpretation is that of balls and bars:

an r-combination of M (k types of objects)	$\longleftrightarrow$	a permutation of r balls and $k-1$ bars

Example: If the bagel shoppe sells plain, poppy-seed, sesame, and everything bagels, in how many ways are there to make a bag of a dozen bagels?

Example: What if there must be at least one of each kind in the dozen?

§3.5 Infinite-Multiset-Combination Examples

Another interpretation of r-combinations of M is *non-negative* integer solutions of

$$x_1 + x_2 + x_3 + x_4 = r.$$

Example: How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

Example: How many positive integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?