§3.1 Counting Principles

Goal: Count the number of objects in a set.

Notation: When S is a set, $|S|$ denotes the number of objects in the set. This is also called S's *cardinality*.

Addition Principle: When you want to count a set which you are able to break down into subsets, then count these parts individually and take their sum.

Mathematically, suppose that S is a set of objects. We say that S_1, S_2, ..., S_k is a *(set) partition* of S if

- $S = S_1 \cup S_2 \cup \cdots \cup S_k$
- $S_i \cap S_j = \emptyset$ for all i and j. **(*)**

Addition Principle: If $S_1, S_2, \ldots, S_k$ is a partition of S, then $|S| = |S_1| + |S_2| + \cdots + |S_k|$.

The addition principle is used to break down a larger set into more manageable pieces.

§3.1 Counting Principles

Example: A student wants to take either a math class or a biology class to keep his workload down. If there are three math classes and four biology classes to choose from, how many choices are there in all?

A: Assuming there is no cross-listed course,

$$3 + 4 = 7.$$

Arrange yourselves in groups of two or three for

Partition Boggle

Example: You are organizing the yogurt section of the store; determine how many types of yogurt:

(a) if there are ten flavors and three styles.

(b) if in addition there are four brands for each.

(c) if in addition there are two sizes each.

§3.1 Multiplication Principle

This is the *Multiplication Principle*: If a first task has p outcomes and a second task has q outcomes <u>*for all outcomes of the first task*</u>, then the two tasks performed successively have pq outcomes.

Multiplication Principle Practice Groupwork:
1) How many 2-digit numbers have non-zero digits?

2) How many two-digit numbers have distinct and non-zero digits?

3) How many odd numbers between 1000 and 9999 have distinct digits? [*Hint: It may be useful to choose the digits in a non-standard way.*]

4) How many poker hands are full houses? [*Poker hands contain five cards; a full house has three cards of one value and two cards of a different value.*]

Another approach to 2):

§3.1 Subtraction Principle

Let A be a set and U be a larger set containing A. Define the *complement* of A in U, written $\overline{A}$ or A^c, as the objects in U not in A. In other words, $A^c = U \setminus A$.

Subtraction Principle:
Let $A \subset U$. Then $|A^c| = |U| - |A|$.

Example: If computer passwords consists of the digits 0–9 and the letters a–z, then how many passwords have a repeated symbol?

Total possibilities	$-$	Distinct-digit possibilities
36^6		$36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31$
2,176,782,336		1,402,410,240

Total: 774,372,096 (About 35% of the total #.)

Division Principle: Let S be partitioned into k parts of the same size. Then

$$k = \frac{|S|}{|\text{part}|}.$$

§3.2–3.3 Permutations and Combinations

The material in Chapter 3 consists of how to count arrangements of objects. Two types:

An r-*permutation* of a set S is an **ordered** arrangement of r of its n elements.

An r-*combination* of a set S is an **unordered** arrangement of r of its n elements.

Consider the set $S = \{a, b, c\}$:

	r-permutation of S	r-combination of S
$r = 1$		
$r = 2$		
$r = 3$		
$r = 4$		

When we discuss permutations of a set S with no reference to an r, then we are arranging **all** of S's elements. Notice: It makes no sense to discuss a combination of a set.

§3.2–3.3 Counting Arrangements

Notation: $n! = n(n-1)(n-2)\cdots 2\cdot 1$.
By convention, $0! = 1$

How many r-permutations are there
of an n-element set?

$$P(n,r) := n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

$P(3,1)=\frac{3!}{2!}=3 \quad P(3,2)=\frac{3!}{1!}=6 \quad P(3,3)=\frac{3!}{0!}=6$

How many r-combinations are there
of an n-element set?
Notation: $C(n,r) = \binom{n}{r}$ "n choose r"

Theorem 3.3.1. $P(n,r) = r!\binom{n}{r}$

Corollary. In factorial notation, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Proof of Theorem 3.3.1

Theorem 3.3.1. $P(n, r) = r!\binom{n}{r}$

Proof: Let S have n elements. Then r-combinations of S and r-permutations of S are related in the following way:

Every r-permutation of S can be generated in *exactly one way* using the following steps:

1. Choose r elements from S.
2. Order the r elements in some way.

There are $\binom{n}{r}$ ways to choose r elements from S, and $r!$ ways to permute these r elements. By the multiplication principle, $P(n, r) = r!\binom{n}{r}$. □

Another proof is by way of the division principle:

Q: How many r-permutations of S contain the exact same elements?
A: $r!$, since r elements can be arranged in $r!$ ways.

If we look at all r-permutations of S and disregard order, then each r-combination of S appears $r!$ times. Therefore, $\binom{n}{r} = \frac{P(n,r)}{r!}$. □

§3.2–3.3 Arrangement Examples

Example: How many 4-letter "words" can be formed from the letters $\{a, b, c, d, e\}$?

Example: In how many ways can __ out of the __ enrolled students attend class?

Example: In how many ways can these __ students seat themselves in the __ chairs?

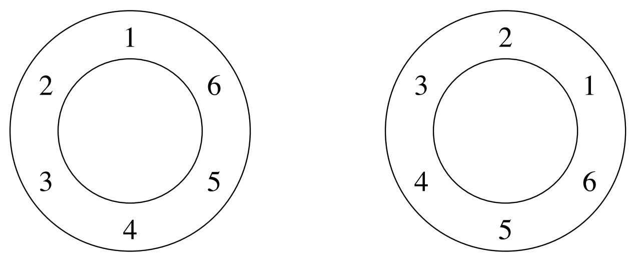
Example: In how many ways could the instructor see __ students in __ chairs?

Example: How many seven-digit numbers are there such that the digits are distinct, taken from $\{1, 2, \ldots, 9\}$, and such that 5 and 6 do not appear consecutively in either order?

§3.2 Circular Permutations

Example: If six children are marching in a circle, how many different ways can they form their circle?

We need to be careful because multiple circular arrangements are equivalent:



This is an example of a *circular permutation*; this is in contrast to the *linear permutations* that we dealt from before.

We can use the division principle to count circular permutations. We notice that there are __ linear permutations for every circular permutation.

Theorem 3.2.2. In general, the number of circular r-permutations of a set of n elements is: