Lecture Notes

Combinatorics

Christopher Hanusa

Queens College

Spring 2009

References:

Bóna, Miklós. A Walk Through Combinatorics.
Brualdi, Richard. Introductory Combinatorics.
Graham, Knuth, and Pat. Concrete Mathematics.

How To Count

Given some discrete objects,
what properties and structures do they have?

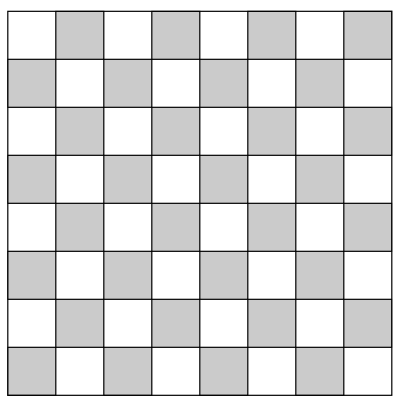
- Can we count the arrangements?
- Can we enumerate the arrangements?
- Do any arrangements have a certain property?
- Can we construct arrangements some property or do there exist optimal arrangements?

§1.1 Chessboard Tilings

Consider an 8×8 chessboard and 32 dominoes.

A *domino* is a tile that covers two adjacent squares on the chessboard.

Q: Is it possible to cover the whole chessboard with the dominoes with no overlap? (A *tiling* of the board.)



Q: In **how many** ways can we tile the chessboard?

A:

§1.1 Chessboard Tilings

Q: How many domino tilings are there of an $m \times n$ board?

A: If m and n are both odd, then zero.
If m and n are both even, then

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

(A formula!)

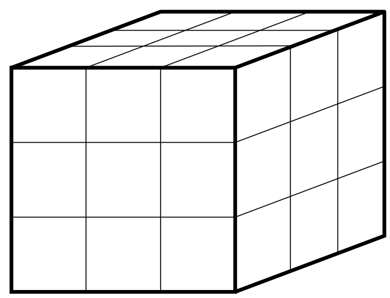
Q: What happens to our 8×8 board if we remove two opposite corners? How many tilings are there of this board?

A:

Proof:

§1.2 Cutting a Cube

Cut a 3×3 cube into twenty-seven 1×1 cubes using as few cuts as possible. (Rearrangements are allowed.)



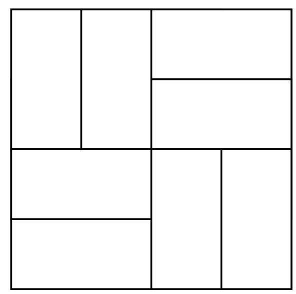
What is the simplest answer? (The most obvious)

Can you do better?

Conjecture:____is the minimum possible number of cuts.
Proof:

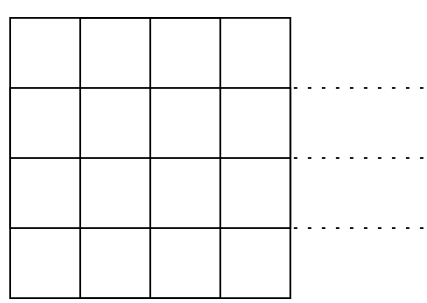
§1.2 Fault Lines in Tilings

Along the same "lines" as chopping a cube:
Which domino tilings have a *fault line*?



Conjecture: In any domino tiling of a 4×4 chessboard, there is a fault line.

Proof: Define x_1, x_2, and x_3 to be the number of dominoes crossing the first, second, and third separators, respectively:



Every vertical domino must intersect exactly one of these separators; we can count the number of vertical dominoes by adding $x_1 + x_2 + x_3$.

§1.2 Fault Lines in Tilings

- There can not be an odd number of dominoes crossing a separator.

- If there exists a 4×4 domino tiling with no fault line, then x_1, x_2, and x_3 are all positive (and therefore ≥ 2)

- Therefore there must be at least six vertical dominoes in any faultless 4×4 domino tiling.

- Similarly, there are $\geq$ six horizontal dominoes

- However, a 4×4 chessboard can hold but 8 dominoes, a contradiction!

Therefore, it is impossible for a 4×4 domino tiling to have no fault lines.

§1.5 Combinatorial Designs

Some of the best combinatorial problems come from recreational mathematics. Here is one from a deck of cards.

Consider the Jack, Queen, and King cards in three suits, $\diamondsuit$, $\heartsuit$, and $\clubsuit$.

Q: Can we take these nine cards are arrange them in a 3×3 array so that there is one card of every suit and one card of every type in each row and column?

K$\diamondsuit$	Q$\heartsuit$	J$\clubsuit$
J$\heartsuit$	K$\clubsuit$	Q$\diamondsuit$
Q$\clubsuit$	J$\diamondsuit$	K$\heartsuit$

Let us recognize that we have two arrays:

3	2	1
1	3	2
2	1	3

1	2	3
2	3	1
3	1	2

In both, each object appears in each row and column with the added restriction that when the arrays are linked, each of the nine pairs occurs.

§1.5 Latin Squares

3	2	1
1	3	2
2	1	3

1	2	3
2	3	1
3	1	2

Each single array is called a *Latin Square*.

When two Latin Squares satisfy the additional restriction that every pair of objects appears, we say that this is a set of *orthogonal* Latin Squares.

Q: When do there exist orthogonal Latin Squares of order n?

A: Yes, unless $n = 2$ or $n = 6$!

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are the only 2×2 Latin Squares; they are not orthogonal.

The $n = 6$ case was asked by Euler in the Eighteenth Century: "Is it possible for 36 officers of six different ranks and from six different regiments to be arranged in a 6×6 formation where each row and column contains one officer of each rank and from each regiment?"

§1.7 The game of Nim

Here are the rules of the two-player game Nim:

1. The game starts with two piles of counters.
2. Alternating play, each player removes some number of counters from **either** pile.
3. The player who removes the last counter wins.

Let's play!

- First, get a feel for the game. Try starting with initial piles of (4,6), (5,5), (3,10), and (7,8).
- Next, start to develop some strategies for winning.
- Finally, determine conditions under which the first player will always win if she plays optimally, and similarly for the second player.

If you finish this before time is up, try playing Nim with three or more initial piles.

§1.7 The game of Nim

Your notes: