Algorithms

Definition: An **algorithm** is a set of rules followed to solve a problem.

In general, an algorithm has the steps: Havel–Hakimi:

1. Organize the input.

2. Repeatedly apply some steps
until a termination condition holds

3. Analyze data upon termination

Computers can be used to run the algorithms once we verify they work.

To verify the **correctness** of an algorithm:

1. Verify that the algorithm terminates. (often invoking finiteness)

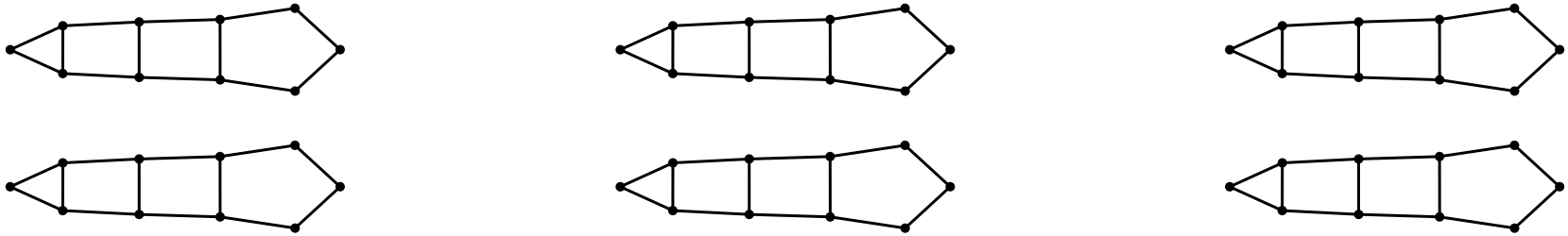
2. Verify that the result satisfies the desired conditions.

Matchings in Graphs

Recall: A **matching** M in a graph G is a subset of edges of G that share no vertices.

Definition: A **maximal matching** M is a matching such that the inclusion into M of any edge of $G \setminus M$ is no longer a matching.

Definition: A **maximum matching** is a matching M that has the most edges possible for the graph G.



Thought Exercise: What is the result of overlapping two matchings?

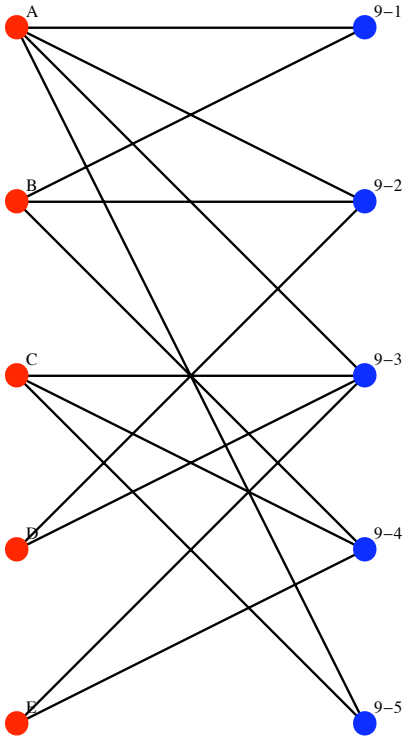
Recall: A **perfect matching** is a matching involving every vertex of G.

$\star$ We will discuss matchings in a bipartite graph $\star$

Application: Scheduling

Suppose you are working in a group trying to complete all the problems on the homework. Depending on everyone's preferences, you would like to assign each member one problem to do.

Person A likes problems 9-1, 9-2, 9-3, and 9-5.
Person B likes problems 9-1, 9-2, and 9-4.
Person C likes problems 9-3, 9-4, and 9-5.
Person D likes problems 9-2 and 9-3.
Person E likes problems 9-3 and 9-4.



Create a graph that models the situation.

Question:
What is a maximum matching for this graph?

We will use an algorithm to answer this question.

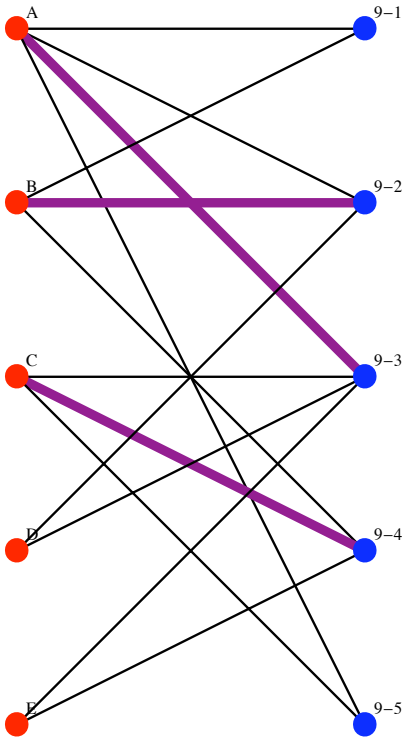
Motivating The Hungarian Algorithm

Let us work through the basic idea behind the algorithm.
We start with an initial matching; we might as well make it maximal.
Why is the pictured matching maximal?

Definition: Given a matching M in a graph G, an M-**alternating path** is a path in G that starts at a vertex not in M, and whose edges alternate between being in M and not in M.

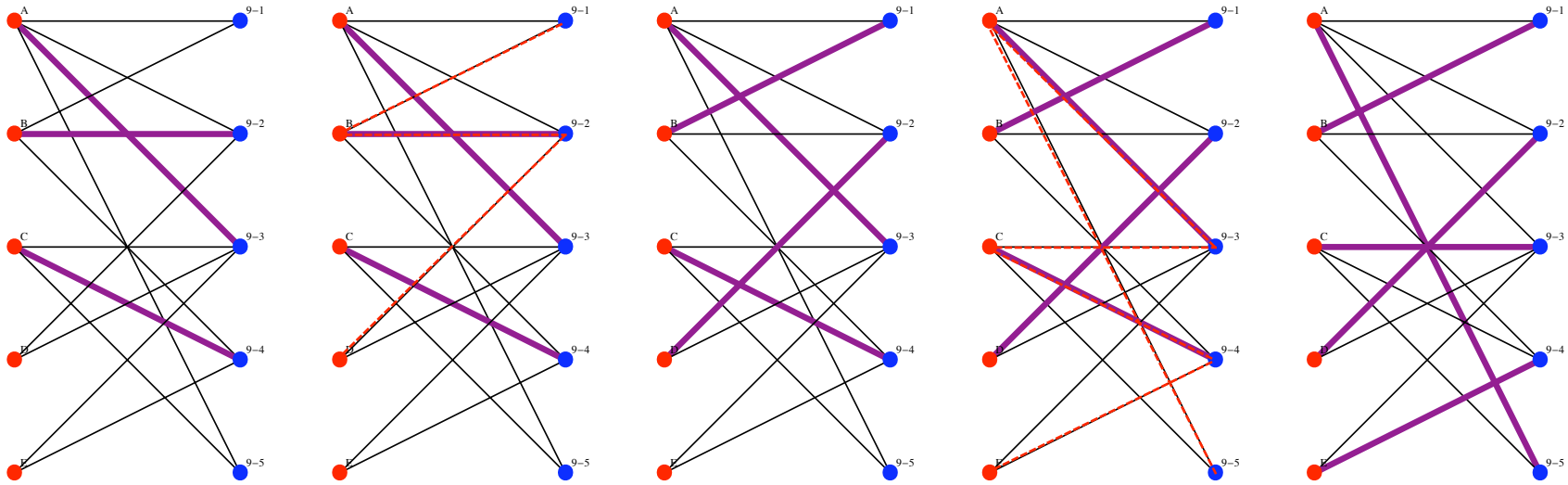
Example: $D \to$ 9-2 $\to B \to$ 9-4 $\to C$ is an M-alternating path.

Definition: An M-**augmenting path** is an M-alternating path that begins AND ends at unmatched vertices.



It is augmenting because we can improve M by toggling the edges between those in M and those not in M.

Motivating The Hungarian Algorithm



Given M, $P = D \longrightarrow$ 9-2 $\longrightarrow B \longrightarrow$ 9-1 is an M-augmenting path. Toggling the edges in P gives a new matching M'.

Given M', $P' = E \longrightarrow$ 9-4 $\longrightarrow C \longrightarrow$ 9-3 $\longrightarrow A \longrightarrow$ 9-5 is an M'-augmenting path. Toggling the edges in P' gives a new matching M''.

The matching M'' is maximal.

The Hungarian Algorithm

The Hungarian Algorithm: (Kuhn, König, Egeváry) [*Finds a maximum matching in a bipartite graph (w/red and blue vertices)*]

1. Start with a bipartite graph G and any matching M. Label all red vertices *eligible* (for augmentation).
2. If all red, eligible vertices are matched, stop. Otherwise, there exists a red, unmatched, eligible vertex to use in the next step.
3. Let v be an unmatched, eligible, red vertex. Start growing all possible M-alternating paths from v. That is, follow every edge not in M to a blue vertex. From a matched blue vertex, follow the edge of M back to a red vertex, and repeat as far as possible.

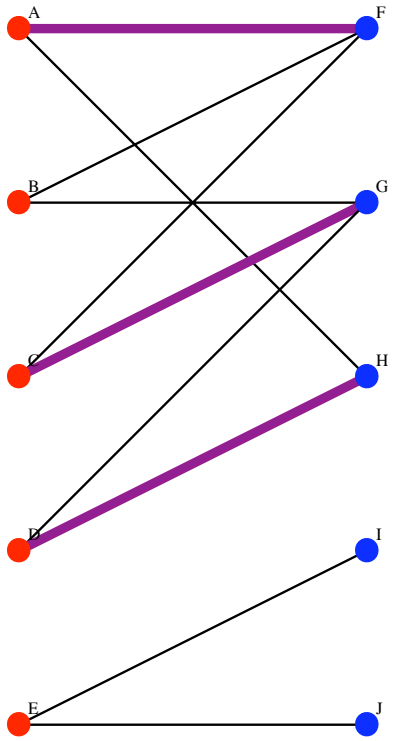
$$\begin{cases} \text{If there is an } M\text{-augmenting path, toggle edges to augment } M. \\ \text{If there is no } M\text{-augmenting path, mark } a \text{ ineligible.} \end{cases}$$

Return to Step 2.

Applying the Hungarian Algorithm

Here is something that might happen during an application of the Hungarian algorithm:

Example: There is no M-augmenting path starting at B in the graph to the right.



We would mark B ineligible and move on to the next eligible, unmatched red vertex in the graph (E).

Proof of Correctness

Claim: The Hungarian Algorithm gives a maximum matching.
Proof: We must show that the algorithm always stops, and that when it stops, the output is indeed a maximum matching.

The algorithm terminates. Each time Step 3 is run, one red vertex either becomes matched or becomes ineligible. Also, no red vertex that starts matched becomes unmatched. Since there are a finite number of red vertices, the algorithm must terminate.

The output is a maximum matching. The output M is a matching inducing no M-augmenting paths in the graph. Suppose that there were another matching M^* that used more edges than M.

When we overlap M and M^*, the result is a union of cycles and paths. At least one path must have more edges from M^* than M.

This path is an M-augmenting path, contradicting the definition of M.