k-Connectivity

Definition: G is k-**connected** if • $|V(G)| > k$, and
• Removing fewer than k vertices does not disconnect the graph.

(We will say that every graph is 0-connected.)

Definition: The **connectivity** of G (denoted $\kappa(G) =$ "kappa") is the maximum k such that G is k-connected.

(Conventions: $\kappa(\{v\}) = 0$ and $\kappa(K_n) = n - 1$.)

k-Connectivity

Definition: A **separating set** (or **vertex cut**) is a set of vertices $X \subset V(G)$ such that $G \setminus X$ is disconnected.

Therefore, $\kappa(G) =$ size of the *minimum* separating set.

Definition: A **cut vertex** is a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected.

$\kappa(G) = 0 \iff G$ is disconnected or G is a single vertex.
$\kappa(G) \geq 2 \iff G$ has no cut vertex.

k-Edge-Connectivity

Definition: G is k-**edge-connected** if

- Removing fewer than k edges does not disconnect the graph.

(We say that every graph is 0-edge-connected.)

Definition: The **edge connectivity** of G (denoted $\lambda(G) =$ "lambda") is the maximum k such that G is k-edge connected.

k-Edge-Connectivity

Definition: A **disconnecting set** is a set of edges $D \subset E(G)$ such that $G \setminus D$ is disconnected.

Therefore, $\lambda(G) =$ size of the *minimum* disconnecting set.

Definition: A **bridge** is an edge $e \in E(G)$ such that $G \setminus e$ is disconnected.

$\lambda(G) = 0 \iff G$ is disconnected or G is a single vertex.
$\lambda(G) \geq 2 \iff G$ has no bridge.

Connectivity Facts

If you delete a cut vertex from a graph, . . .

If you delete a bridge from a graph, . . .

Theorem 2.4.1 Let G be connected. Then

$$G \text{ is a tree } \iff \text{ Every edge of } G \text{ is a bridge.}$$

Theorem 3.2.1 A regular graph of even degree has no bridge.

For all graphs G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Minimum vs. Minimal

Here we have hit upon an important concept— the difference between minim*um* and minim*al*.

Minimum refers to an element of <u>absolute</u> smallest size.
Minimal refers to an element of <u>relative</u> smallest size.

Example: minimum vs. minimal disconnecting set:

Example: maximum vs. maximal connected subgraph:

Blocks

Definition: A **block** of a graph G is a maximally connected subgraph of G with no cut vertex.

The following things are true about blocks.

1. G itself may be a block.
2. Except for blocks that are edges, blocks are always 2-connected.
3. Any two blocks share at most one vertex.
4. A vertex shared between blocks is a cut vertex of G.
5. The blocks of G partition $E(G)$.

Characterization of 2-connectivity

(Whitney, 1932) Let G be a graph with at least 3 vertices. Then,

G is 2-connected $\iff$ for all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.

(Menger, 1932) Let G be a graph with at least $k+1$ vertices. Then,

G is k-connected $\iff$ for all $v, w \in V(G)$, there exist k internally disjoint v, w-paths in G.

Definition: Let H be any subgraph of G. Then an H-**path** (or an **ear**) is a path in G that starts and ends in H.
Definition: Given a graph G, an **ear decomposition** of G is a sequential construction of G starting with some cycle C, and at each step successively adding to the existing graph H some H-path.

Characterization of 2-connectivity

Let G be a graph with ≥ 3 vertices. The following are equivalent:

1. G is 2-connected.
2. G is connected and has no cut vertex.
3. G is a block.
4. For all $v, w \in V(G)$, there exist two internally disjoint v, w-paths in G.
5. For all $v, w \in V(G)$, there exists a cycle in G through v and w.
6. $\delta(G) > 0$ and for all $e, f \in E(G)$, there exists a cycle in G through e and f.
7. G has an ear decomposition.