Maximal Planar Graphs

A graph with “too many” edges isn’t planar; how many is too many?

Goal: Find a numerical characterization of “too many”

Definition: A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

Examples: Octahedron K_4 $K_5 \setminus e$

What do we notice about these graphs?

Numerical Conditions on Planar Graphs

- Every face of a maximal planar graph is a triangle!

Theorem 8.1.2: If G is maximal planar, then $q = 3p - 6$.

Proof: In any plane drawing of G, let $p = \#$ of vertices,
$q = \#$ of edges, and $r = \#$ of regions.
We will count the number of face-edge incidences in two ways.

From a face-centric POV, the number of face-edge incidences is

From an edge-centric POV, the number of face-edge incidences is

Substitute into Euler's formula:

- Every planar graph is a subgraph of a maximal planar graph.
- Every maximal planar graph has exactly $q = 3p - 6$ edges.

Cor 8.1.3: Every planar graph with p vertices has at most $3p - 6$ edges!

Numerical Conditions on Planar Graphs

Theorem 8.1.4: The graph K_5 is not planar.

Proof:

Theorem 8.1.5:* If G is planar with girth ≥ 4, then $q \leq 2p - 4$.

Proof: Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

Theorem 8.1.5: If G is planar and bipartite, then $q \leq 2p - 4$.

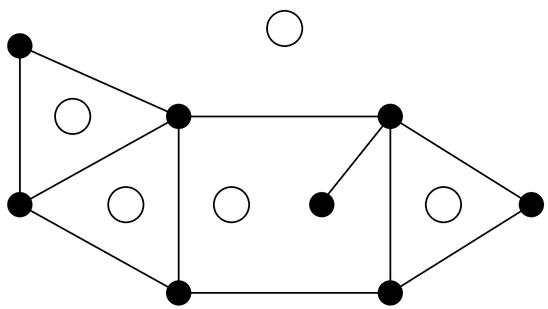
Theorem 8.1.6: $K_{3,3}$ is not planar.

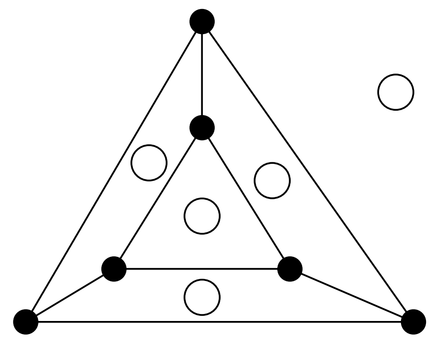
Theorem 8.1.7: Every planar graph has a vertex with degree ≤ 5.

Proof:

Dual Graphs

Definition: Given a plane drawing of a planar graph G, the **dual graph** $D(G)$ of G is a graph with vertices corresponding to the regions of G. Two vertices are connected by an edge each time the two regions share an edge as a border.



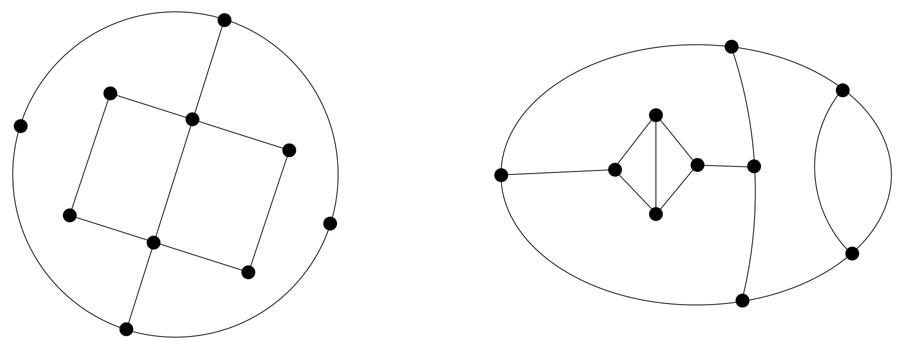


- The dual graph of a simple graph may not be simple.
  - Two regions may be adjacent multiple times.
- G and $D(G)$ have the same number of edges.

Definition: A graph G is **self-dual** if G is isomorphic to $D(G)$.

Maps

Definition: A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a **normal map**.



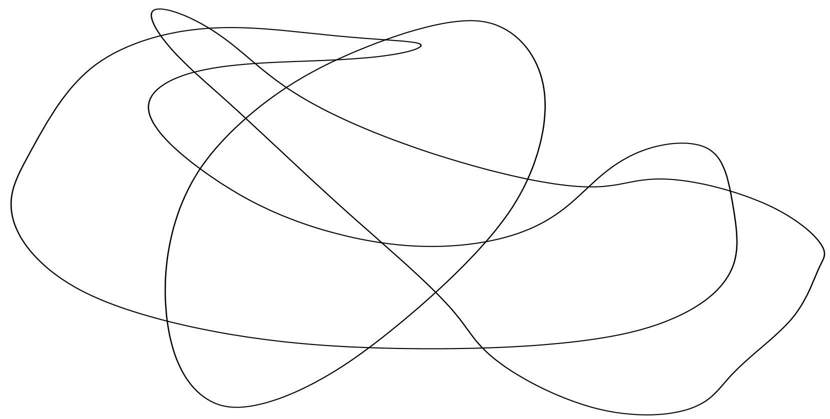
Definition: In a map, the regions are called **countries**. Countries may share several edges.

Definition: A **proper coloring** of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

Question: How many colors are necessary to properly color a map?

Proper Map Colorings

Lemma 8.2.2: If M is a map that is a union of simple closed curves, the regions can be colored by two colors.



Proof: Color the regions R of M as follows:

$$\begin{Bmatrix} \text{black} & \text{if } R \text{ is enclosed in an odd number of curves} \\ \text{white} & \text{if } R \text{ is enclosed in an even number of curves} \end{Bmatrix}.$$

This is a proper coloring of M. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.

The Four Color Theorem

Lemma 8.2.6: (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

Proof: Very hard.

$\star$ This is the wrong object $\star$

Theorem: If G is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

- Every face of G is a triangle $\rightsquigarrow$
- G is connected $\rightsquigarrow$
- G is planar $\rightsquigarrow$

The Four Color Theorem

Assuming Lemma 8.2.6,

$$\begin{aligned} G \text{ is maximal planar} &\Rightarrow D(G) \text{ is a normal map} \\ &\Rightarrow \text{countries of } D(G) \text{ 4-colorable} \\ &\Rightarrow \text{vertices of } G \text{ 4-colorable} \\ &\Rightarrow \chi(G) \leq 4 \end{aligned}$$

This proves

Theorem 8.2.8: If G is maximal planar, then $\chi(G) \leq 4$.

Since every planar graph is a subgraph of a maximal planar graph, Lemma C implies:

Theorem 8.2.9: If G is a planar graph, then $\chi(G) \leq 4$.

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