Planarity

Up until now, graphs have been completely abstract.
In Topological Graph Theory, it matters how the graphs are drawn.

- Do the edges cross?
- Are there knots in the graph structure?

Definition: A **drawing** of a graph G is a pictorial representation of G in the plane as points and line segments. The line segments must be **simple curves**, which means no intersections are allowed.

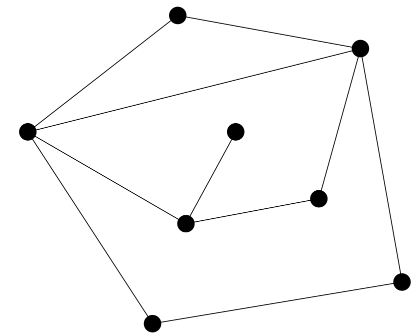
Definition: A **plane drawing** of a graph G is a drawing of the graph in the plane with no crossings. Otherwise, G is **nonplanar**.

Definition: A **planar graph** is a graph that has a plane drawing.

Example: K_4 is a planar graph because
is a plane drawing of K_4.

Vertices, Edges, and Faces

Definition: In a plane drawing, edges divide the plane into **regions**, or **faces**.



There will always be one face with infinite area. This is called the **outside face**.

Notation: Let $p = \#$ of vertices, $q = \#$ of edges, and $r = \#$ of regions. Compute the following data:

Graph	p	q	r	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

In 1750, Euler noticed that ____________ in each of these examples.

Euler's Formula

Theorem 8.1.1: (Euler's Formula) If G is connected, then in a plane drawing of G, $p - q + r = 2$.

Proof: (by induction on the number of cycles)
Base Case: If G is a tree, there is one region, so

$$p - q + r = p - (p - 1) + 1 = 2.$$

Inductive Step: Suppose that for all plane drawings with fewer than k cycles, $p - q + r = 2$, we wish to prove that in a plane drawing of a graph G with exactly k cycles, $p - q + r = 2$ also holds.

Let C be a cycle in G. Let e be any edge in C, then e is adjacent to two different regions, one inside C and one outside C.

$G \setminus e$ has fewer cycles than G, and one fewer region. The inductive hypothesis holds for $G \setminus e$, giving