Characterization of Graphs with Eulerian Circuits

There is a simple way to determine if a graph has an Eulerian circuit.

Theorems 3.1.1 and 3.1.2: Let G be a pseudograph that is connected* *except possibly for isolated vertices*. Then,
G has an Eulerian circuit $\iff$ the degree of every vertex is even.

The Königsberg bridge pseudograph has four vertices of odd degree, and therefore does not have an Eulerian circuit.

($\Rightarrow$) Euler, 1736: Given an Eulerian circuit C, each time a vertex appears in the circuit, there must be an "in edge" and an "out edge", so the total degree of each vertex must be even.

($\Leftarrow$) Hierholzer, 1873: This is harder; we need the following lemma.

Proof of Lemma 3.1.3

Lemma 3.1.3: If the degree of every vertex in a pseudograph is even, then every non-isolated vertex lies in some circuit in G.

Proof: Start a trail at any non-isolated vertex A in G.

Whenever the trail arrives at some other vertex B, there must be an odd number of edges incident to B not yet traversed by the trail.

So there is some edge to follow out of B; take it.

The trail must eventually return to A, giving us a circuit.

Proof of Theorem 3.1.2

$\star$ Each vertex in G has even degree $\Longrightarrow$ G has an Eulerian circuit $\star$

Find the longest circuit C in G. If C uses every edge, we are done. If not, we'll show a contradiction to the maximality of C.

Remove all edges of C from G and any isolated vertices to form H.
H is a pseudograph where each vertex is of even degree.
Since G is connected, C and H must share a vertex A.
Write C as $C = \cdots e_1 A e_2 \cdots$.

Find a circuit D in H through A.
Write D as $D = \cdots f_1 A f_2 \cdots$.
No edges of D repeat nor are they in C
Define a new circuit $C' = \cdots e_1 A f_2 \cdots f_1 A e_2 \cdots$.

C' is a longer circuit in G than C, contradicting its maximality. $\square$

Other related theorems

Theorem 3.1.6: Let G be a connected* pseudograph. Then, G has an Eulerian trail $\Leftrightarrow$ G has exactly two vertices of odd degree.

Proof: Let x and y be the two vertices of odd degree. Add edge xy to G; $G + xy$ is a pseudograph with each vertex of even degree. By Theorem 3.1.2, there exists an Eulerian circuit in $G + xy$. Remove xy from the circuit and you have an Eulerian trail in G.

Remark: When drawing a picture without lifting your pencil, start and end at the vertices of odd degree!

Theorem 3.1.5: A pseudograph G has a decomposition into cycles if and only if every vertex has even degree.

Application: de Bruijn sequences

Consider the following example of a **de Bruijn sequence**:

0000110101111001

Each of the sixteen binary sequences of length 4 are present (where we allow cycling):

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

This is the most compact way to represent these sixteen sequences.

Sequence definitions

Definition: An **alphabet** is a set $\mathcal{A} = \{a_1, \ldots, a_k\}$.
Definition: A **sequence** or **word** from $\mathcal{A}$ is a succession
$S = s_1 s_2 s_3 \cdots s_l$, where each $s_i \in \mathcal{A}$; l is the **length** of S.

Definition: A sequence is called a **binary sequence** when $\mathcal{A} = \{0, 1\}$.

Definition: A **de Bruijn sequence** of order n on the alphabet $\mathcal{A}$ is a sequence of length k^n such that every word of length n is a consecutive subsequence of S. (and called **binary** if $\mathcal{A} = \{0, 1\}$)

Theorem: A de Bruijn sequence of any order n on any alphabet $\mathcal{A}$ always exists.

Proof: Use the theory of Eulerian circuits on certain graphs:

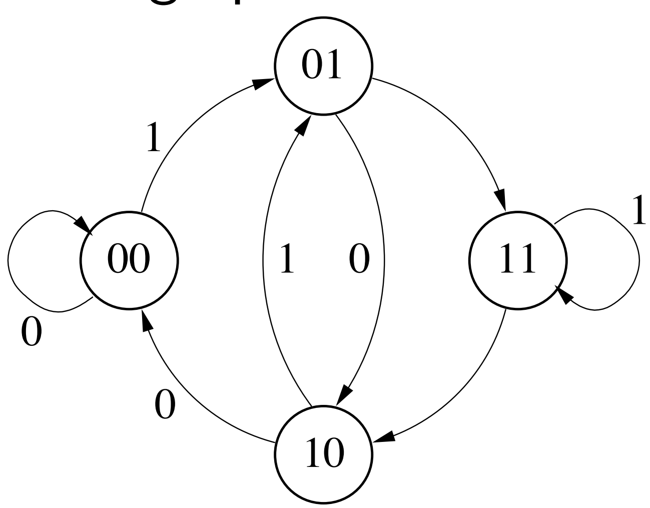
de Bruijn graphs

Definition: The **de Bruijn graph** of order n on $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ is a directed pseudograph that has as its vertices words of $\mathcal{A}$ of length $n-1$. Each vertex has k out-edges corresponding to the k letters of the alphabet $\mathcal{A}$. Following edge a_i adds letter a_i to the end of the sequence and removes the first letter from the sequence:

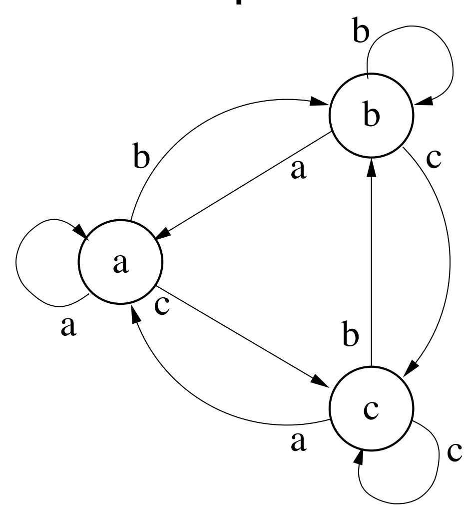
$$\boxed{b_1 b_2 \cdots b_{n-1}} \xrightarrow{a_i} \boxed{b_2 \cdots b_{n-1} a_i}$$

Examples:

The binary de Bruijn graph of order 3



The de Bruijn graph of order 2 on the alphabet $\mathcal{A} = \{a, b, c\}$.



Proof that a de Bruijn sequence always exists

The de Bruijn graph G of order n on alphabet $\mathcal{A}$ is connected and each vertex has as many edges entering as leaving the vertex. This implies that G has an Eulerian circuit C (of length k^n).

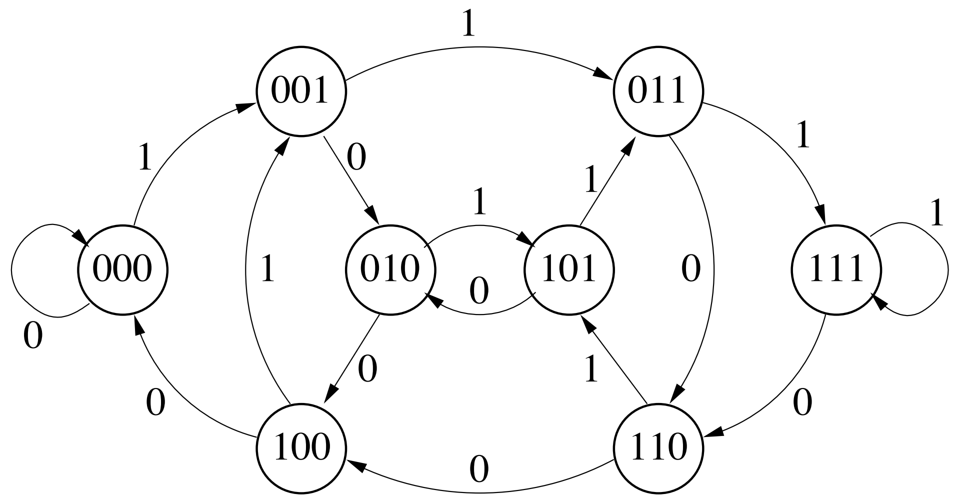
Follow C and record in order the sequence S of labels of edges visited.

Claim: S is a de Bruijn sequence of order n on $\mathcal{A}$.

We know that S is of length k^n. We must now verify that every sequence of length n appears as a consecutive subsequence in S.

By construction, the sequence of the $n-1$ labels of edges visited before arriving at a vertex is exactly the name of the vertex. The word formed by this name followed by the label of an outgoing edge is a word of $\mathcal{A}$ of length n and is different for every edge of C. This implies that every sequence appears as a consecutive subseq. of S.

Example: The binary de Bruijn graph of order 4



1. Find an Eulerian circuit in this graph.
2. Write down the corresponding sequence.
3. Verify that it is a de Bruijn sequence. (use chart, p.77)
4. Convince yourself that the name of a vertex is the same as the sequence formed by the three previous edges.