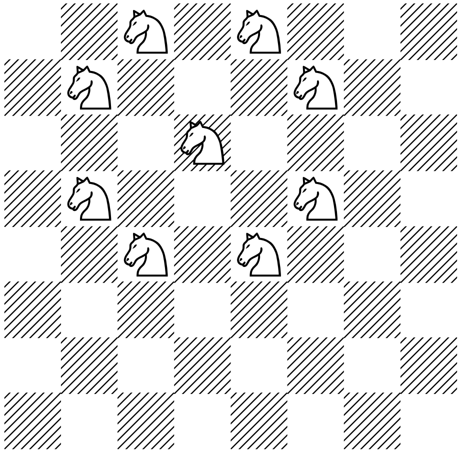
Knight's Tours

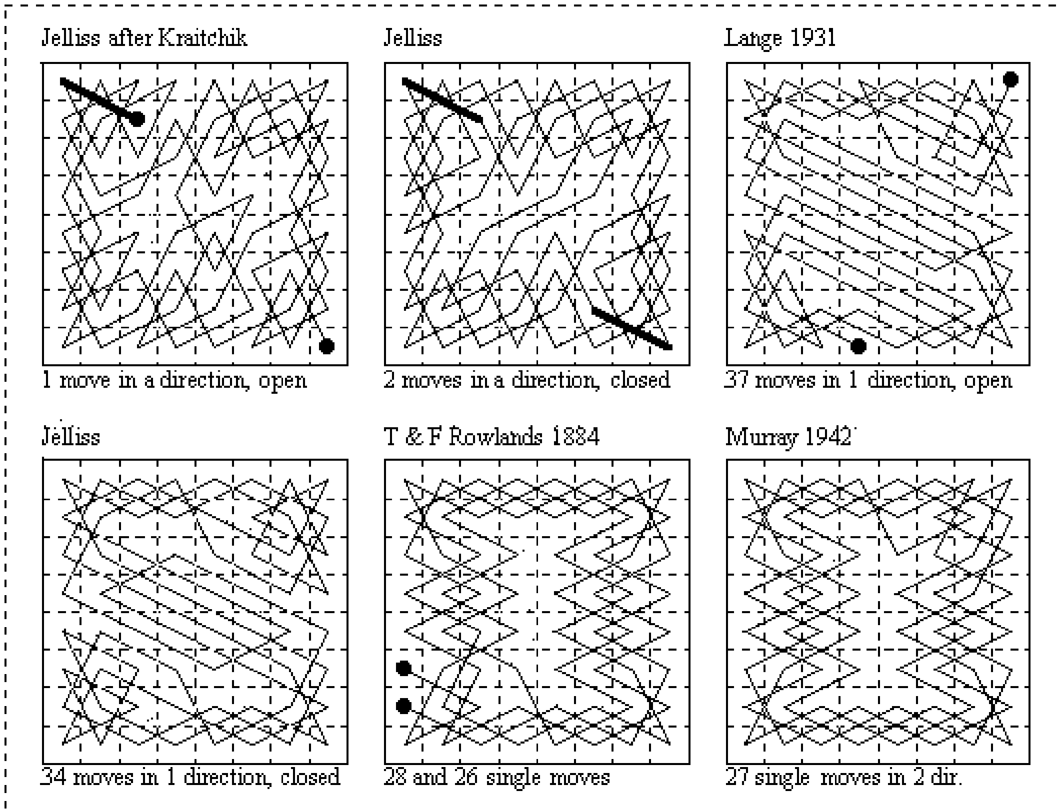
In chess, a knight (♘) is a piece that moves in an "L": two spaces over and one space to the side.



Question: Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an 8×8 chessboard once? (How about return to where it started?)

Definition: A path of the first kind is called an **open knight's tour**. A cycle of the second kind is called a **closed knight's tour**.

8 × 8 Knight's Tour

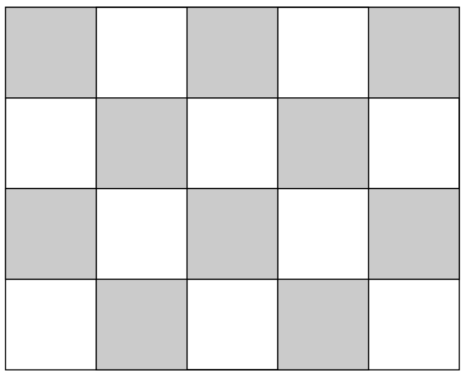


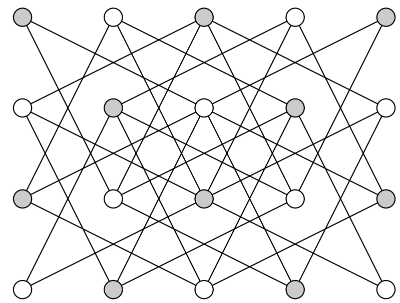
Source: `http://www.ktn.freeuk.com/ga.htm`

Question: Are there any knight's tours on an $m \times n$ chessboard?

The Graph Theory of Knight's Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight's move away.





An open/closed knight's tour
on the board

A knight move always alternates between white and black squares. Therefore, a knight move graph is always _______________.

Question: Are there any knight's tours on an $m \times n$ chessboard?

Knight's Tour Theorem

Theorem: An $m \times n$ chessboard with $m \leq n$ has a *closed* knight's tour unless one or more of these conditions holds:

1. m and n are both odd.
2. $m = 1$, 2, or 4.
3. $m = 3$ and $n = 4$, 6, or 8.

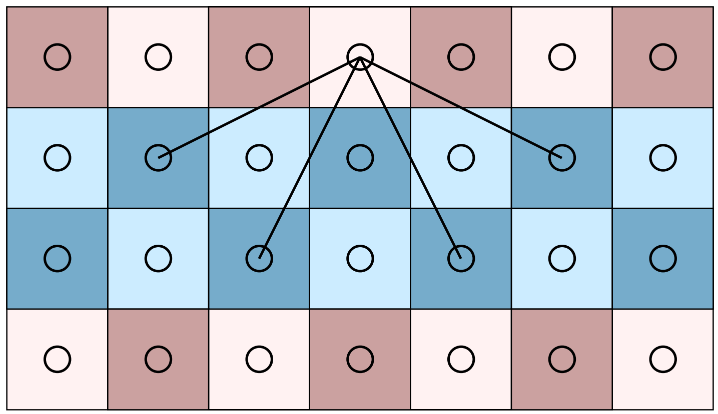
"Proof": We will only show that it is impossible in these cases.

Case 1: When m and n are both odd,

Case 2: When $m = 1$ or 2, the knight move graph is not connected.

Knight's Tour Theorem

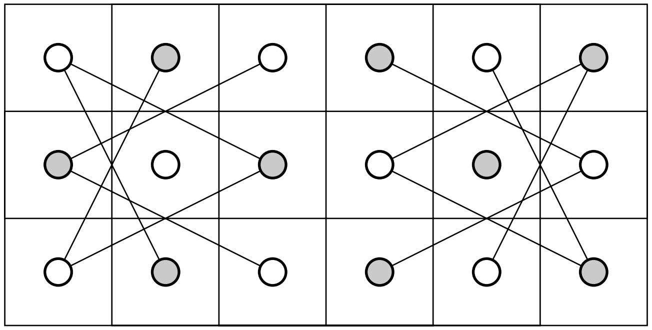
Case 2: When $m = 4$, consider:



Suppose we can find a Hamiltonian cycle C in the graph.
Then, C must alternate between white and black vertices.
Also, every red vertex in C is adjacent to only blue vertices.
There are the same number of red and blue vertices.
So, C must alternate between red and blue vertices. This means:
All vertices of C are "white and red" or "black and blue".

Knight's Tour Theorem

Case 3: 3×4 is covered by Case 2. Consider the 3×6 board:



Assume that there is a Hamiltonian cycle C in G.
Then, C visits each vertex v and uses two of v's incident edges.
If $\deg(v) = 2$, then both of v's incident edges are in C.
Draw in all these "forced edges" above. With just these forced edges, there is already a cycle of length four. This cycle cannot be a subgraph of any Hamiltonian cycle, contradicting its existence. $\square$

The 3×8 case is similar, and part of your homework.

See also: "Knight's Tours on a Torus", by J. J. Watkins, R. L. Hoenigman

The Origins of Graph Theory

City of Königsberg in 1736

Is it possible to start out anywhere, cross all seven bridges exactly once, and return to where you started?

We can model this situation with a graph:

Question: Can we draw this graph without lifting our pencil?

Pseudographs

This is not a graph—it's a pseudograph. For this section and others, we will allow *multiple edges* and *loops*. A few of our definitions need updating.

Types of "walks" in pseudographs:

Repeat Vertices?	Repeat Edges?	Open $A_1 \neq A_n$	Closed $A_1 = A_n$
No	No	path	cycle
Yes	No	trail	circuit
Yes	Yes	walk	closed walk

Definition: The **length** of a "walk" is the number of *edges* involved.

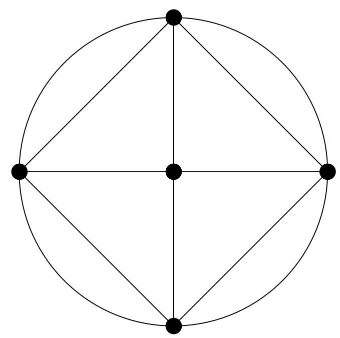
Remark: In a simple graph, the smallest cycle possible is of length 3. In a pseudograph, there may exist cycles of length 1 and 2.

Definition: The **degree** of a vertex v is the number of edges incident with v; loops count twice!

Eulerian Circuits

Definition: An **Eulerian circuit** C in a graph G is a circuit containing every edge of G.
Definition: An **Eulerian trail** T in a graph G is a trail containing every edge of G.

$\star$ Important: Trails and circuits do not use any edge twice. $\star$



So the Königsberg bridge problem in the language of graph theory is:

Is there an Eulerian circuit in the corresponding pseudograph?