Bipartite graphs

Question: What is $\chi(C_n)$ when n is odd?

Answer:

Definition: A graph is called **bipartite** if $\chi(G) \leq 2$.

Examples: $K_{m,n}$, $\square_n$, Trees

Theorem 2.1.6: G is bipartite $\Longleftrightarrow$ every cycle in G has even length.

($\Rightarrow$) Let G be bipartite. Assume that there is some cycle C of odd length contained in G . . .

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

Suppose not. Then there are two vertices v and w of the same color that are adjacent. This generates a contradiction because there exists an odd cycle as follows:

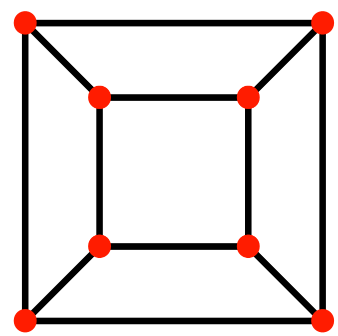
Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition: An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \longrightarrow \{1, 2, \ldots, l\}$.]

Definition: A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example: Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

Definition: The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ "chi prime".

Edge coloring theorems

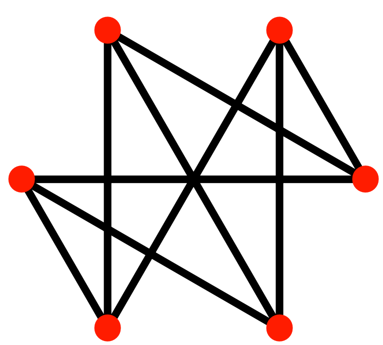
Theorem 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

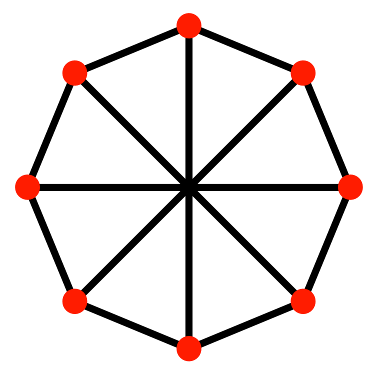
Theorem 2.2.2: Vizing's Theorem:
For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G)+1$.

Proof: Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Fact: **Most** 3-regular graphs have edge chromatic number 3.

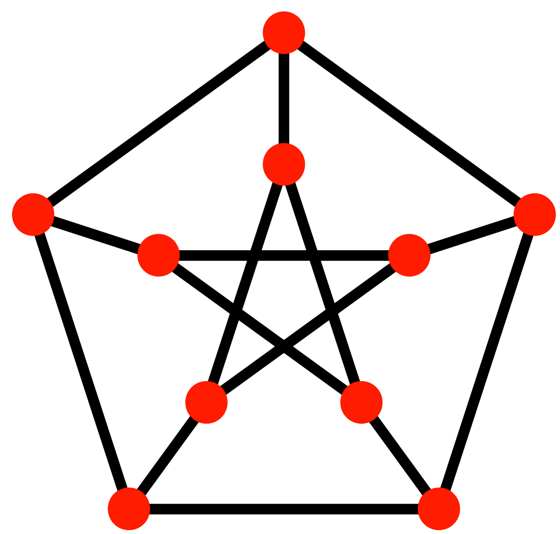




Snarks

Definition: A 3-regular graph with edge chromatic number 4 is called a **snark**.

Example: The Petersen graph P:



The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$____ or ____.

If $\chi'(K_n) =$____, then each vertex has an edge leaving of each color.

Q: How many red edges are there?

This is only an integer when:

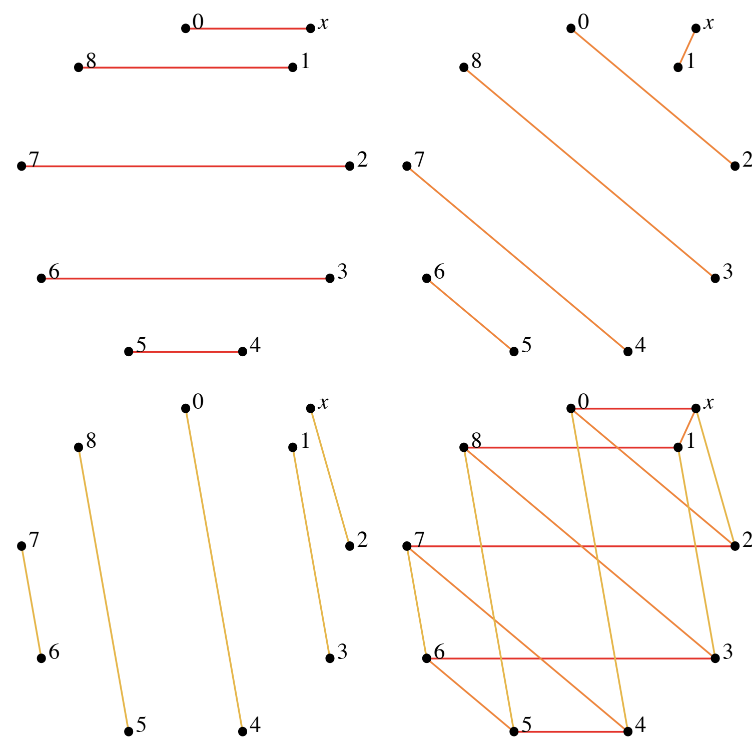
So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

The edge chromatic number of complete graphs

Theorem 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof: We prove this using the *turning trick*.

Label the vertices of K_{2n}
$0, 1, \ldots, 2n-2, x$. Now,
Connect 0 with x ,
Connect 1 with $2n-2$,
⋮
Connect $n-1$ with n.



Now **turn** the edges.
And do it again. (and again, . . .)

Each time, new edges are used.
This is because each of the edges is a different "circular length": vertices are at circ. distance 1, 3, 5, . . ., 4, 2 from each other, and x is connected to a different vertex each time.

The edge chromatic number of complete graphs

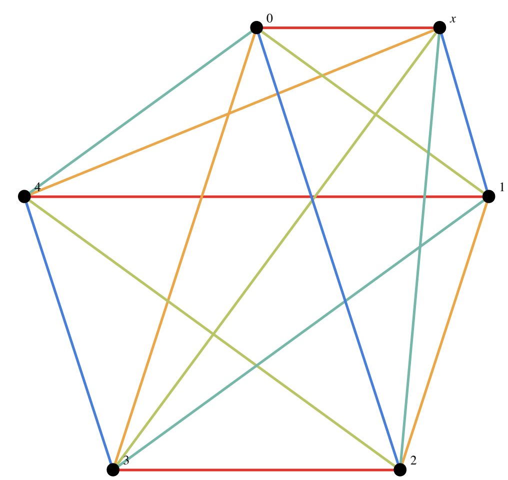
Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n - 1$ colors—simply delete vertex x!

This is related to the area of combinatorial designs.

Question: Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.