(Vertex) Colorings

Today we will discuss the idea of vertex colorings.

Definition: A **coloring** of a graph G is a labeling of the vertices of G with colors. [Technically, it is a function $f : V(G) \to \{1, 2, \ldots, l\}$.]

Definition: A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled with the same color.

Example: W_6:

We can properly color W_6 with ____ colors and no fewer.

Of interest: What is the fewest colors necessary to properly color G?

The chromatic number of a graph

Definition: The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example: $\chi(K_n) =$ ____

Proof: In order to have a proper coloring of K_n, we would need to use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

⋆ $\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors. (Show it!)
2. There is no proper coloring of G with $k - 1$ colors. (Prove it!)

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof: If $\chi(G) = k$, then there is a proper coloring of G using k colors. Let the vertices of H inherit their coloring from G. This gives a proper coloring of H using k colors, which implies $\chi(H) \leq k$.

Corollary: For any graph G, $\chi(G) \geq \omega(G)$.

Proof: Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Example: Calculate $\chi(G)$ for this graph G:

Critical graphs

One way to prove that G can not be properly colored with $k-1$ colors is to find a subgraph H of G that requires k colors.

How small can this subgraph be?

Definition: A graph G is called **critical** if for every proper subgraph $H \subsetneq G$, then $\chi(H) < \chi(G)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

Proof: If G is critical, stop. Define $H = G$.
If not, then there exists a proper subgraph G_1 of G with ________.
If G_1 is critical, stop. Define $H = G_1$.
If not, then there exists a proper subgraph G_2 of G with ______$\cdots$
Since G is finite, there will be some proper subgraph G_l of G such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

Critical graphs

What do we know about critical graphs?

Theorem 2.1.1: Every critical graph is connected.

Theorem 2.1.3: If G is critical with $\chi(G) = 4$, then for all $v \in V(G)$, $\deg(v) \geq 3$.

Proof by contradiction: Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.