Connectivity

Definition: A graph G is **connected** if for every pair of vertices a and b in G, there is a **path from a to b in G**. That is, there exists a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ such that $v_0 = a$, $v_k = b$, and $v_{i-1}v_i$ is an edge of G for all i, $1 \leq i \leq k$.

When a graph is **disconnected**, it does have connected subgraphs.

Definition: In a graph G, a **connected component** H is a maximally connected subgraph. (In terms of vertices and edges.)

More precisely, for every subgraph K such that

$$H \subsetneqq K \subset G,$$

K is not connected.

Lemmas A and B

Lemma A: If there is a path from vertex a to vertex b in G and from vertex b to vertex c in G, then there is a path from vertex a to vertex c in G.

Proof: By hypothesis,

- There exist paths $P : av_1v_2 \cdots v_kb$ and $Q : bw_1w_2 \cdots w_lc$ in G.
- If all the vertices are distinct, path R :
- If not all vertices are distinct,

Lemma B: Let G be a connected graph. Let C be a cycle contained in G. Let e be any edge in C. If H is the graph resulting from removing e from G, then H is connected.

Proof: You will prove this in the homework.

Hints:

Connectivity and edges

Theorem 1.3.1: If G is a connected graph with p vertices and q edges, then $p \leq q + 1$.

Proof: Induction on the number of edges of G.

- **Base Case.** If G is connected and has fewer than three edges, then G equals either:

- **Inductive Step.**
  Inductive hypothesis:
  $p \leq q + 1$ holds for all connected graphs with

  We want to show:
  $p \leq q + 1$ holds for all connected graphs with

Break into cases, depending on whether G contains a cycle.

Connectivity and edges

- **Case 1.** There is a cycle C in G.

Use Lemma B. After removing an edge from C, the resulting graph H is connected...

- **Case 2.** There is no cycle in G.

Find a path P in G that can not be extended.

Claim: The endpoints of P, a and b, are leaves of G.

Remove a and its incident edge to form a new graph H.
If possible, apply the inductive hypothesis to H.

□

⋆ Important Induction Item: Always **remove** edges. ⋆

Trees and forests

Definition: A **tree** is a connected graph that contains no cycles.
Definition: A **forest** is a graph that contains no cycle.

These definitions imply: (Fill in the blanks)

1. Every connected component of a forest ________________.
2. A connected forest ________________.
3. A subgraph of a forest ________________.
4. A subgraph of a tree ________________.
5. Every tree is a forest.

Thm 1.3.2, 1.3.3: Let G be a connected graph with p vertices and q edges. Then, G is a tree $\iff p = q + 1$.

Thm 1.3.5: G is a tree iff there exists exactly one path between each pair of vertices.

Proof Sketches

Proof of Thm 1.3.3: [G connected. Then $p = q + 1 \Rightarrow G$ is a tree.]

Proof by contradiction. Suppose G is connected and not a tree.
We want to show that $p \neq q + 1$.

Proof of Thm 1.3.5: [G tree iff exactly one path between v_1 and v_2.]

Suppose that G is a tree. Then at least one path between v_1 and v_2.
What if two paths, $P_1 = v_1 u_1 u_2 \cdots u_n v_2$ and $P_2 = v_1 w_1 w_2 \cdots w_m v_2$?

($\Leftarrow$) Suppose that G is not a tree. Then if G is not connected,
If G is connected and contains a cycle,