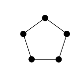
When are two graphs the same?

Two graphs G_1 and G_2 are **equal** ($G_1 = G_2$) if they have the exact same vertex sets and edge sets.
The graphs G_1 and G_2 are **isomorphic** ($G_1 \approx G_2$) if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ such that
$v_i v_j$ is an edge of G_1 iff $\varphi(v_i)\varphi(v_j)$ is an edge of G_2.

In this course, we will spend a large amount of time trying to figure out whether two given graphs are the same.

Side note: For a graph, the set of homomorphisms (isomorphisms into itself) is a measure of the symmetry of the graph. Ex. 

Simple operations on graphs

The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ can mean two different things:

- When the vertex sets are different, the **(disjoint) union** H of G_1 and G_2 is formed by placing the graphs side by side. In this case, $H = (V_1 \cup V_2, E_1 \cup E_2)$.
- When the vertex sets are the same, then the **union** H of G_1 and G_2 contains every edge of both E_1 and E_2. In this case, $H = (V, E_1 \cup E_2)$.

The **complement** G^c or $\overline{G}$ of a graph $G = (V, E)$ is a graph with the same vertex set. Its edge set contains all edges **NOT** in G.

If $G = (V, E_1)$ and $G^c = (V, E_2)$, then
$E_1 \cup E_2 = E(K_n)$, and $E_1 \cap E_2 = \emptyset$.

Subgraphs

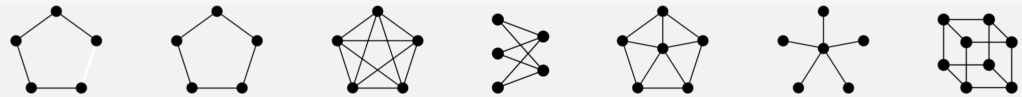
A **subgraph** H of a graph G is a graph where every vertex of H is a vertex of G, and that every edge of H is an edge of G.
★ If edge e of G is in H, then the endpoints of e must also be in H.

If G_1 and G_2 are two graphs, we say that G_1 **contains** G_2 if there exists a subgraph H of G_1 such that H is isomorphic to G_2.

An **induced subgraph** H of a graph G is determined by a set of vertices $W \subseteq V(G)$. Define H to have as its vertex set, W, and as its edge set, the set of edges from $E(G)$ between vertices in W.

Induced subgraphs of G are always subgraphs of G, but not vice versa.

Families of Graphs

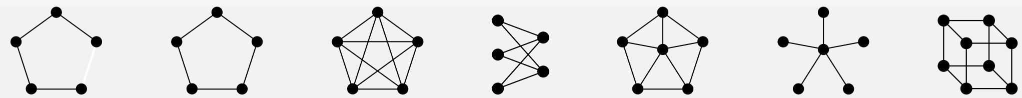


- **Path graph** P_n: The path graph P_n has $n+1$ vertices, $V = \{v_0, v_1, \ldots, v_n\}$ and n edges, $E = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$.
  - ★ The **length** of a path is the number of *edges* in the path.

- **Cycle graph** C_n: The cycle graph C_n has n vertices, $V = \{v_1, \ldots, v_n\}$ and n edges, $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

We often try to find and/or count paths and cycles in a graph.

Families of Graphs

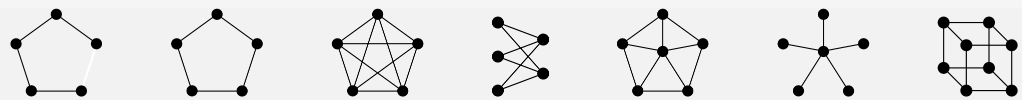


- **Complete graph** K_n: The complete graph K_n has n edges, $V = \{v_1, \ldots, v_n\}$ and has an edge connecting every pair of distinct vertices.

Definition: a **bipartite** graph is a graph where the vertex set can be broken into two parts such that there are no edges between vertices in the same part.

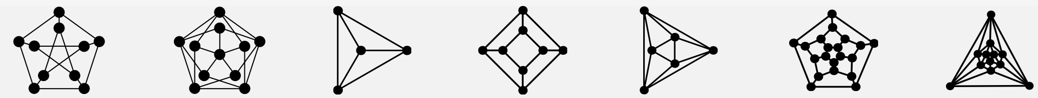
- **Complete bipartite graph** $K_{m,n}$: The complete bipartite graph $K_{m,n}$ has $m + n$ vertices $V = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ and an edge connecting each v vertex to each w vertex.

Families of Graphs



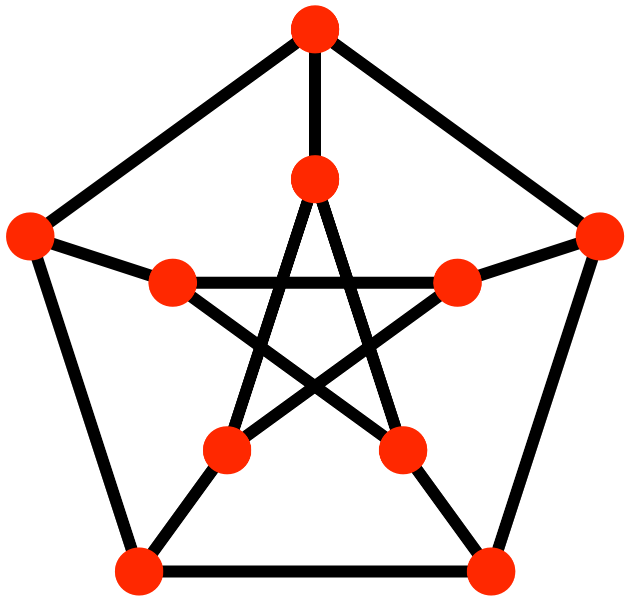
- **Wheel graph** W_n: The wheel graph W_n has $n+1$ vertices $V = \{v_0, v_1, \ldots, v_n\}$. Arrange and connect the last n vertices in a cycle (the rim of the wheel). Place v_0 in the center (the hub), and connect it to every other vertex.
- **Star graph** St_n: The star graph St_n has $n+1$ vertices $V = \{v_0, v_1, \ldots, v_n\}$ and n edges $\{v_0v_1, v_0v_2, \ldots, v_0v_n\}$.
- **Cube graph** $\square_n$: The cube graph in n dimensions, $\square_n$, has 2^n vertices. We index the vertices by binary numbers of length n. We connect two vertices when their binary numbers differ by exactly one digit.

Special Graphs

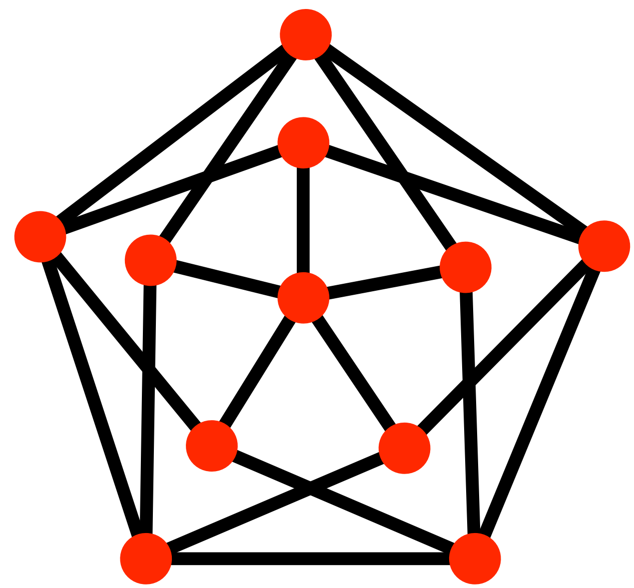


Two graphs we will see on a consistant basis are:

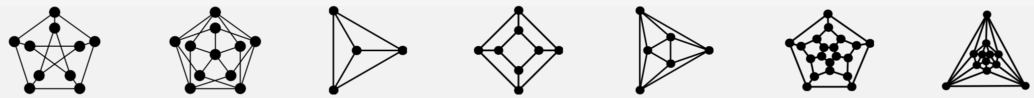
Petersen graph *P*



Grötzsch graph *Gr*



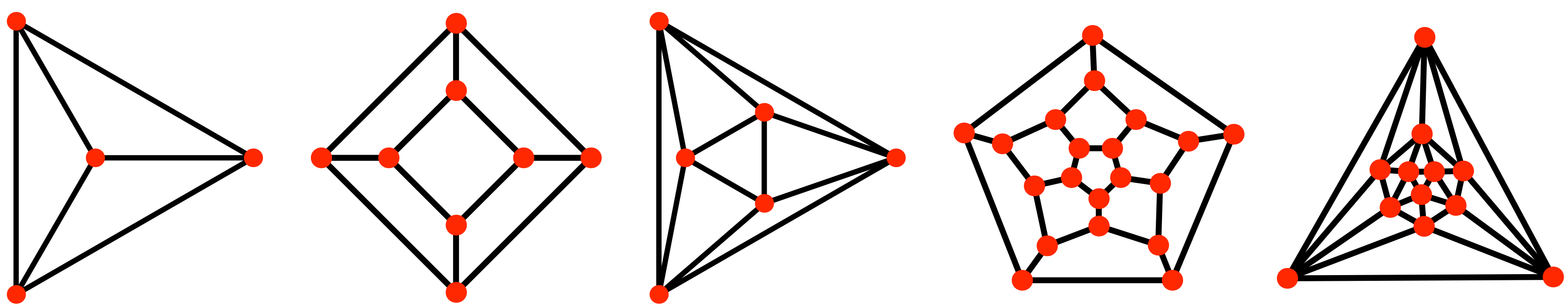
Special Graphs



Definition: The **platonic solids** are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. They are the only regular convex polyhedra made of regular polygons.

Definition: The **Schlegel diagram** of a polyhedron is a planar 2D graph that represents a 3D object, where vertices of the graph represent vertices of the polyhedron, and edges of the graph represent the edges of the polyhedron.

- The **Platonic graphs** are the Schlegel diagrams of the five platonic solids.



Groupwork