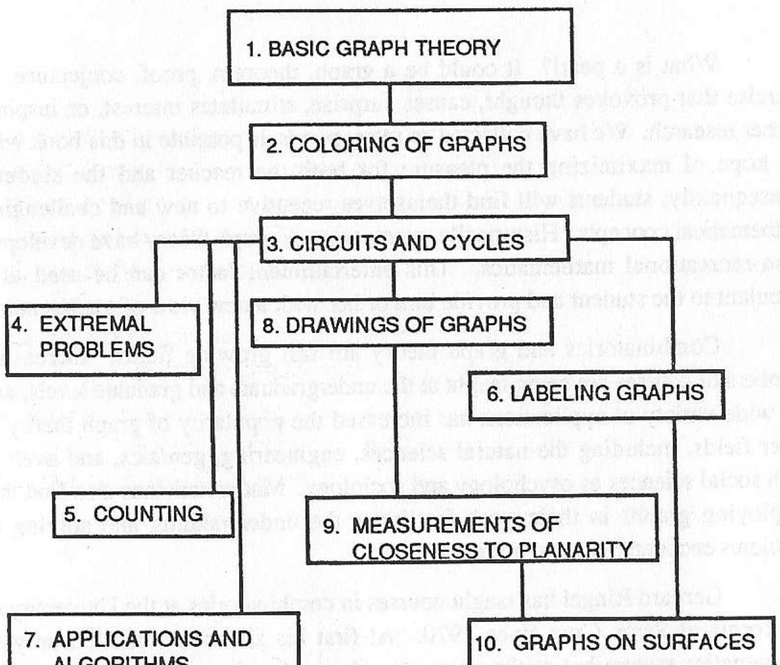
Prerequisites for this book are a strong interest in the subject and a good foundation in high school mathematics. A course in discrete mathematics is recommended but not necessary.

The exercises following each section have been designed to implement the concepts of the section and, we hope, to motivate further study. The starred exercises are considered particularly challenging.

The following chart explains the relationship among the chapters of the book. Chapters 1, 2, and 3 are considered basic to the understanding of the remainder of the text. For a more mathematically intensive course, we suggest that Chapters 6, 8, 9, and 10 be studied. If there are computer science majors in the class, Chapter 7 would be of prime importance. If time is available, Chapters 4 and/or 5 are recommended. The authors believe that Chapter 8 is the most interesting and should not be omitted from any course.



Chapter 1

BASIC GRAPH THEORY

1.1 Graphs and Degrees of Vertices

Before we give any definitions and theory, let us consider the following example. There is a basket containing an apple, a banana, a cherry and a date. Four children named Erica, Frank, Greg and Hank are each to be given a piece of the fruit. Erica likes cherries and dates; Frank likes apples and cherries; Greg likes bananas and cherries; and Hank likes apples, bananas, and dates. Figure 1.1.1 describes the situation. The problem is to give each child a piece of fruit that he or she likes. The reader is invited to find a solution. One can see the advantage of using Figure 1.1.1 as an aid to solving the problem.

Figure 1.1.1 is, in fact, an example of a *graph*. Another example with which we are all familiar is a road map. The map in Figure 1.1.2 is greatly simplified, of course. It shows some different ways of driving from San Jose to San Francisco.

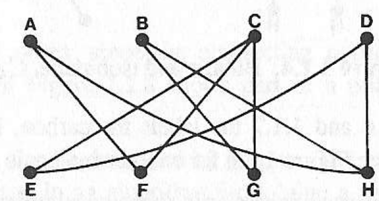


Figure 1.1.1

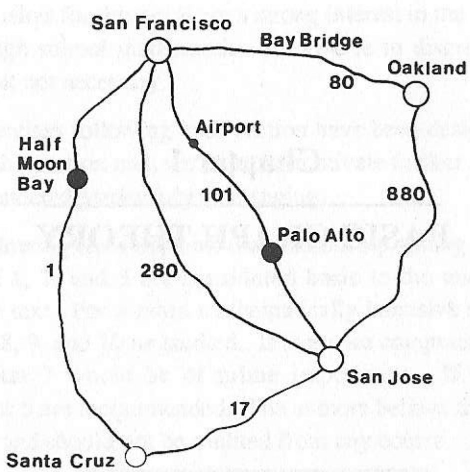


Figure 1.1.2

Chemists use diagrams to picture molecules, and these diagrams are also graphs. Usually the hydrogen atoms are omitted from the diagrams by the chemists using shorthand structure, but the Kekulé structure includes them.

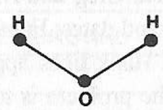


Figure 1.1.3. Water, H_2O.

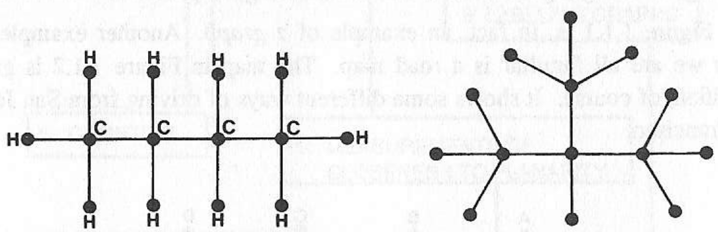


Figure 1.1.4. Butane and isobutane, C_4H_{10}.

In Figures 1.1.6 and 1.1.7 the labels for carbon, hydrogen, and oxygen have been left out. See Figure 1.3.6 for another molecule C_{60}.

Graph theory is used in designing printed circuits for use in electronics devices. Circuits printed on silicon chips must be designed differently from

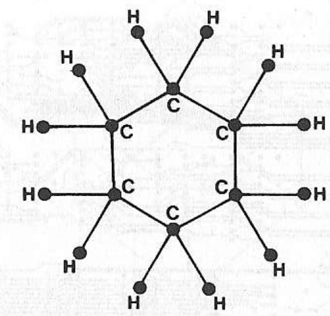


Figure 1.1.5. Cyclohexane, C_6H_{12}.

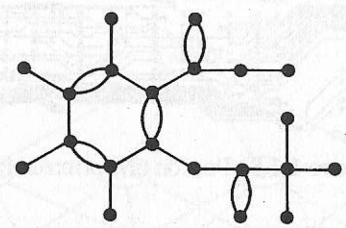


Figure 1.1.6. Aspirin, $C_9H_8O_4$.

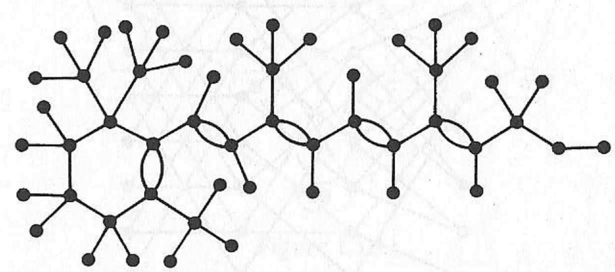


Figure 1.1.7. Vitamin A, $C_{20}H_{30}O$.

those using insulated wires, since the conducting portions cannot cross one another. The graph of Figure 1.1.8 shows part of a printed circuit used in a computer.

Algorithms can also be described by graphs. The graph of Figure 1.1.9 is a chart of the steps used in an algorithm for solving a certain problem using a computer.

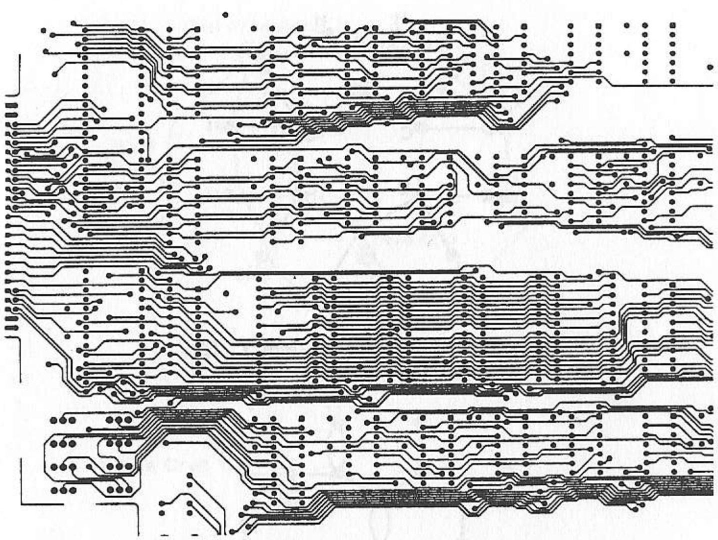


Figure 1.1.8. Portion of a printed circuit

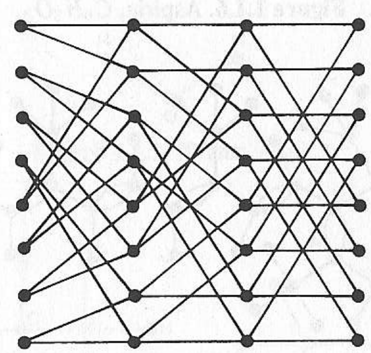


Figure 1.1.9

In the study of lattices and Boolean algebras, graphs arise as diagrams of these structures. Those who have studied set theory will recognize the diagrams of Figures 1.1.10 and 1.1.11 as lattices of subsets. For convenience, we use shorter notation for sets; for example, we write *ABD* for the set $\{A, B, D\}$. The

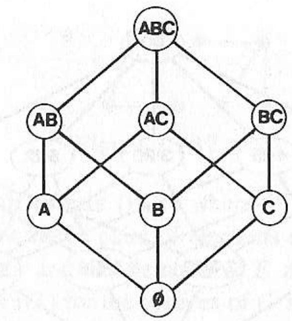


Figure 1.1.10. The lattice of subsets of *ABC*.

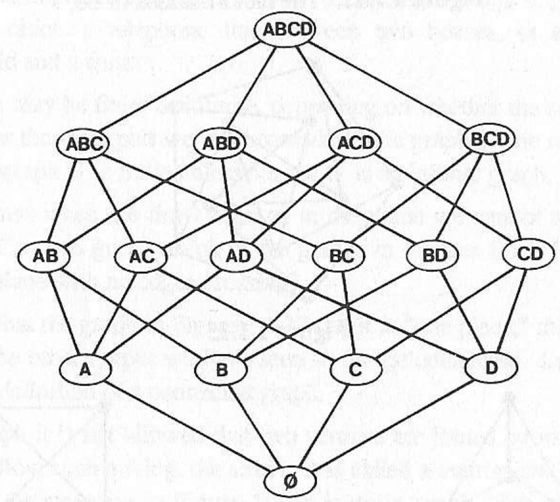


Figure 1.1.11

diagrams of Figures 1.1.10 and 1.1.11 show all subsets of the set at the top. If one set is derived from another by deleting one element, then the two sets are connected by a line in the diagram.

The different factorizations of the integer 60 are shown in the diagram of Figure 1.1.12. Some more examples of graphs are shown in Figures 1.1.13, 1.1.14, 1.1.15, 1.1.16, and 1.1.17.

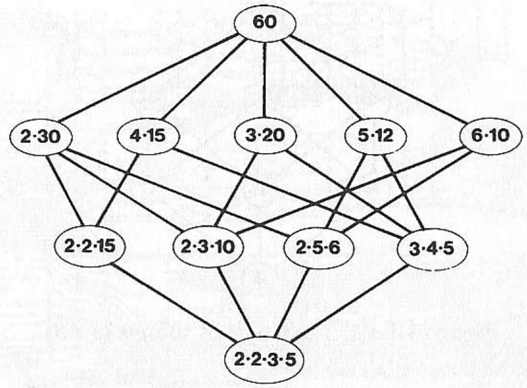


Figure 1.1.12. The factorizations of 60.

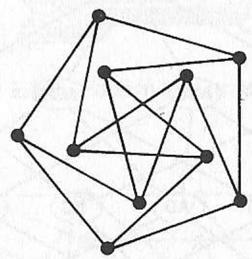


Figure 1.1.13

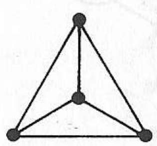


Figure 1.1.14

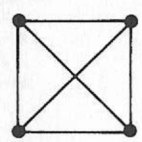


Figure 1.1.15

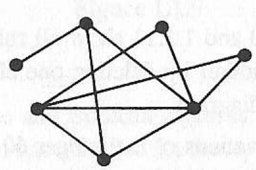


Figure 1.1.16

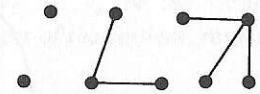


Figure 1.1.17

A *graph* G is a pair of sets (V,E) where V is nonempty, and E is a (possibly empty) set of unordered pairs of elements of V. The elements of V are called the *vertices* of G and the elements of E are called the *edges* of G. Sometimes we will write $V(G)$ for the vertices of G and $E(G)$ for the edges of G. Usually we represent the vertices by points in the plane, but the vertices can be any objects, as in the first example, where the vertices are fruit and children. Similarly, an edge is usually represented by a line connecting two vertices in the plane. The line might be straight or curved. However, an edge might be a road between two cities, a telephone line between two houses, or a preference between a child and a fruit.

A graph may be finite or infinite, depending on whether the set V is finite or infinite. For the most part we shall consider finite graphs. The reader should assume that a graph G is finite unless we say G is an infinite graph.

Sometimes when we draw a graph in the plane we cannot avoid having edges cross. Can you guess which of the graphs in Figures 1.1.1-1.1.17 can be drawn in the plane with no edges crossing?

Notice that the graph in Figure 1.1.17 is not in "one piece," that is, it is not connected. The other graphs we have seen so far are connected. Later we shall give an exact definition of a connected graph.

In a graph it is not allowed that two vertices are joined by more than one edge. If we allow such a thing, the structure is called a *multigraph,* as in Figure 1.1.18. Thus the structure in Figure 1.1.18 is not a graph. The one in Figure 1.1.19 has what we call a *loop,* which is an edge incident with only one vertex. If we allow loops, we call the structure a *pseudograph.* Most of the time we will work only with graphs. So every graph is a multigraph, but not conversely, and every multigraph is a pseudograph, but not conversely. The graphs for aspirin and vitamin A, Figures 1.1.6 and 1.1.7, are examples of multigraphs.

If x and y are vertices of a graph G, we say x is *adjacent* to y if there is an edge between x and y. We shall denote such an edge by xy. We may also say x and y are *neighbors.* We say that a vertex x is *incident* with an edge e if x is an endpoint of e. We also say that e is incident with x whenever x is an endpoint of e.

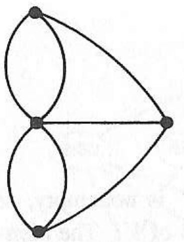


Figure 1.1.18

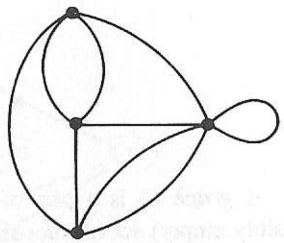


Figure 1.1.19

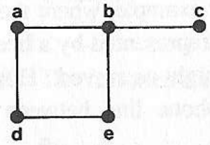


Figure 1.1.20

In Figure 1.1.20, a is adjacent to b and d; b is adjacent to a, c, and e; c is adjacent to b; e is adjacent to b and d; and d is adjacent to a and e. The vertex a is incident with the edges ab and ad; b is incident with the edges ab, bc, and be; and so on.

We normally denote the number of vertices in a graph G by p and the number of edges in G by q.

Look again at Figures 1.1.13, 1.1.14, and 1.1.15. Every vertex is incident with three edges. We say that each vertex has degree 3. Thus the *degree* of a vertex v is the number of edges incident with v. The graph in Figure 1.1.17 has two vertices of degree 0. We call such a vertex an *isolated vertex.* A vertex of degree 1 is called an *end vertex.*

Let us add the degrees of the vertices in the graph shown in Figure 1.1.16.

$$5+4+3+3+2+2+1=20.$$

Now let us count the edges in the graph. The number of edges is $q = 10$. So in this particular graph, when we add up the degrees of all the vertices, the answer we get is twice the number of edges. This should not be a big surprise because when we add the degrees of all the vertices we are adding the number of edges incident with each vertex. But we have added each edge twice since each edge is incident with two vertices. We have just proved

Theorem 1.1.1. *Let $v_1, v_2, \ldots, v_p$ be the vertices of a graph G, and let $d_1, d_2, \ldots, d_p$ be the degrees of the vertices, respectively. Let q be the number of edges of G. Then*

$$d_1 + d_2 + \cdots + d_p = 2q.$$

Theorem 1.1.1 could be stated as follows.

$$\sum_{i=1}^{p} d_i = 2q \quad or \quad \sum_{v \in V} deg\ v = 2q.$$

The Σ notation is used to denote a sum.

Example. Is there a graph with seven vertices, each of which has degree 5? We can answer this question by using Theorem 1.1.1. If there is such a graph, then the sum of all the degrees equals five times seven, which is 35. By Theorem 1.1.1, this number must be an even number, so no such graph exists.

Go back to Figure 1.1.16. We list all the degrees of the vertices

5, 4, 3, 3, 2, 2, 1.

We can do the same for Figure 1.1.17.

3, 2, 1, 1, 1, 1, 1, 0, 0.

These sequences of non-negative integers are the degree sequences of graphs, namely the graphs in Figures 1.1.16 and 1.1.17. A sequence of non-negative integers is called *graphic* if there exists a graph whose degree sequence is precisely that sequence. Consider the problem, which of the following sequences are graphic:

i) 5, 4, 3, 2, 2, 1
ii) 5, 5, 4, 4, 0
iii) 6, 5, 5, 4, 3, 3, 2, 2, 2
iv) 6, 6, 6, 6, 4, 3, 3, 0.

For i) we see immediately that the sum is an odd number, so sequence i) is not graphic. For sequence ii) the sum is even, so we must do something else. Notice that the first number is a five. This means that if there is a graph with this sequence, it has a vertex which is adjacent to five other vertices. But then the graph must have at least six vertices, and there are only five numbers in the sequence. Thus ii) is not graphic. Sequences iii) and iv) cannot be decided that quickly. However, there is an algorithm that works for every sequence. The basic idea behind the algorithm is that one can tell if a given sequence is graphic by looking at another, simpler sequence. In the case of sequence iii) we get the following sequences; how we get them we shall explain later.

(1) 6 5 5 4 3 3 2 2 2
(2) 4 4 3 2 2 1 2 2
(3) 4 4 3 2 2 2 2 1
(4) 3 2 1 1 2 2 1
(5) 3 2 2 2 1 1 1
(6) 1 1 1 1 1 1

Sequence (6) is graphic. It is the degree sequence of the graph in Figure 1.1.21.

Figure 1.1.21

To get sequence (2) from sequence (1), we omit the first number, and since it is a 6, we subtract 1 from each of the next six numbers. To get sequence (3) from sequence (2) we rearrange in descending order. From sequence (3) to sequence (4) we again omit the first number, which is now a 4, then we subtract 1 from each of the next four numbers. We rearrange the order to obtain (5). We omit the first number, which is a 3 and subtract 1 from each of the next three numbers to obtain sequence (6) which is graphic, so we stop. The rule will be that since we obtain a graphic sequence by these steps, then the given sequence (1) is also graphic. Theorem 1.1.2 guarantees that this works.

Theorem 1.1.2. (Havel, Hakimi) *Consider the following two sequences and assume sequence (1) is in descending order.*

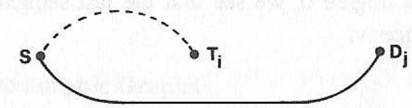
(1) $s, \quad t_1, \quad t_2, \ldots, \quad t_s, d_1, \ldots, d_n$

(2) $\quad t_1-1, t_2-1, \ldots, t_s-1, d_1, \ldots, d_n$

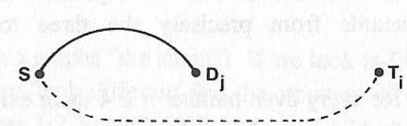
Then sequence (1) is graphic if and only if sequence (2) is graphic.

Proof. First assume that sequence (2) is graphic. Then there is a graph G_2 with degree sequence equal to sequence (2). We construct G_1 from G_2, by adding a vertex and connecting it to the vertices whose degrees are $t_1-1, t_2-1, \ldots, t_s-1$. Then G_1 is a graph with degree sequence equal to sequence (1), hence sequence (1) is graphic.

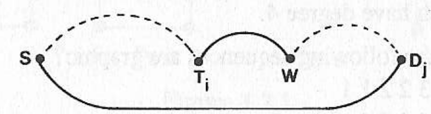
Now we suppose that sequence (1) is graphic. Then there is a graph H, with degree sequence (1). Let the vertex S have degree s, T_i have degree t_i, D_i have degree d_i. If S is adjacent to $T_1, T_2, \ldots, T_s$, then we simply remove this vertex and the edges incident with it, thus obtaining a graph with degree sequence equal to sequence (2). However, this may not be the case. Suppose that for some i with $1 \le i \le s$ vertex S is not adjacent to vertex T_i. Then vertex S is adjacent to some vertex D_j with $j \ge 1$.



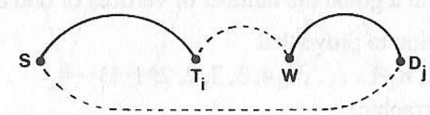
Since the sequence is arranged in descending order, $t_i \ge d_j$. If $t_i = d_j$, we simply exchange T_i and D_j.



If $t_i > d_j$, then T_i has more neighbors than D_j has, so there is a vertex W such that T_i is adjacent to W and D_j is not adjacent to W. We do not know whether W is one of the T-vertices or one of the D-vertices, but it does not matter.



In this case we remove the edges SD_j and T_iW and add the edges ST_i and D_jW to obtain the graph H_2, still having the degree sequence (1).



If S is now adjacent to $T_1, T_2, \ldots, T_s$, as before we remove S to obtain a graph with sequence (2). If not we repeat the above procedure. Since this is a finite graph, for some m, H_m is the graph we seek. Thus sequence (2) is graphic. ■

We now use the procedure to determine whether sequence iv) is graphic.

iv. 6 6 6 6 4 3 3 0
5 5 5 3 2 2 0
4 4 2 1 1 0
3 1 0 0 0

Since there is no graph with one vertex of degree 3, one vertex of degree 1 and three vertices of degree 0, we see that the last sequence is not graphic, hence neither is sequence iv.

Exercises 1.1

1.1.1. Seven students go on vacations. They decide that each will send a postcard to three of the others. Is it possible that every student receives postcards from precisely the three to whom he sent postcards?

1.1.2. a. Prove that for every even number $n \geq 4$ there exists a graph with n vertices, all of which have degree 3, without using Theorem 1.1.2.

b. Prove that for every odd number $n \geq 5$ there exists a graph with $n+1$ vertices such that exactly n vertices have degree 3.

1.1.3. Prove that for every number $n \geq 5$ there exists a graph with n vertices, all of which have degree 4.

1.1.4. Which of the following sequences are graphic?
a. 5 5 4 4 3 2 2 1 1
b. 6 5 4 3 2 2 2 2
c. 4 4 4 4 3 3
d. 7 6 5 4 4 3 2 1

1.1.5. Show that in a graph the number of vertices of odd degree is even.

1.1.6. Use induction to prove that
$(n, n, n-1, n-1, \ldots, 4, 4, 3, 3, 2, 2, 1, 1)$
is always graphic.

1.1.7. Prove that no graph has all degrees different; that is, prove that in a degree sequence there is at least one repeated number.

1.1.8. Is either of the following sequences graphic?
a) (6, 6, 5, 5, 2, 2, 2, 2)
b) (6, 6, 5, 5, 3, 3, 3, 3).

1.1.9. Draw a graph with degree sequence
a) (4, 3, 2, 2, 1)
b) (4, 3, 3, 3, 1).

1.1.10. Figure 1.1.12 shows the multiplication diagram for the number 60. Construct the multiplication diagram for the number 48.

1.1.11. Draw a graph with eight vertices, four of which have degree 4 and four of which have degree 3.

1.2 Subgraphs, Isomorphic Graphs

A *subgraph* of a graph G is a graph H such that every vertex of H is a vertex of G, and every edge of H is an edge of G also. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Figure 1.2.1 shows a graph G with some of its subgraphs.

When are two graphs "the same"? If we look at Figure 1.1.14 and Figure 1.1.15, the drawings look different, but the structure of the graphs is the same. The graphs in Figure 1.2.2 are also the same.

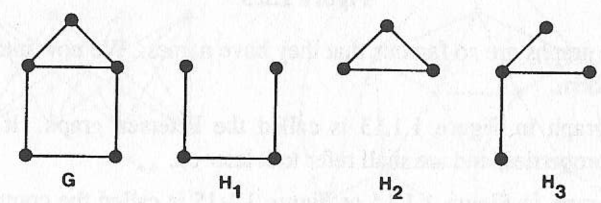


Figure 1.2.1

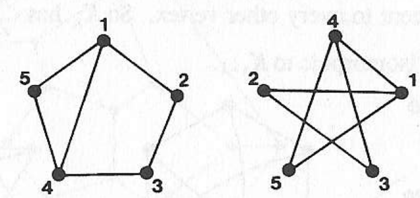


Figure 1.2.2

Two graphs G_1 and G_2 with p vertices are said to be *isomorphic* if the vertices of G_1 and G_2 can be labeled with the numbers from 1 to p such that whenever vertex i is adjacent to vertex j in G_1, then vertex i is adjacent to vertex j in G_2 and conversely. Such a labeling is the same as a one-to-one correspondence between $V(G_1)$ and $V(G_2)$ that preserves adjacency.

In general, for large graphs it is difficult to tell whether two graphs are isomorphic. If two graphs are isomorphic, then they must have the same degree sequence. Thus, if we are asked to determine if two graphs are isomorphic, the first thing we should do is check whether they have the same degree sequence. If the sequences are the same, then we have to do some more checking. For example, the two graphs in Figure 1.2.3 have the same degree sequence, yet they are not isomorphic.

In the graph on the right, each vertex of degree 4 is adjacent to two other vertices of degree 4. In the graph on the left, each vertex of degree 4 is adjacent to only one other vertex of degree 4. Thus they are not isomorphic.

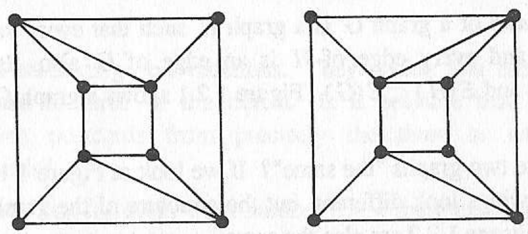


Figure 1.2.3

Some graphs are so famous that they have names. We now introduce you to some of them.

The graph in Figure 1.1.13 is called the Petersen graph. It has many interesting properties, and we shall refer to it later on.

The graph in Figure 1.1.14 or Figure 1.1.15 is called the complete graph on four vertices, denoted K_4. In general, K_n is the graph with n vertices, and every vertex is adjacent to every other vertex. So K_n has $\binom{n}{2} = \dfrac{n(n-1)}{2}$ edges. K_n has n subgraphs isomorphic to K_{n-1}.

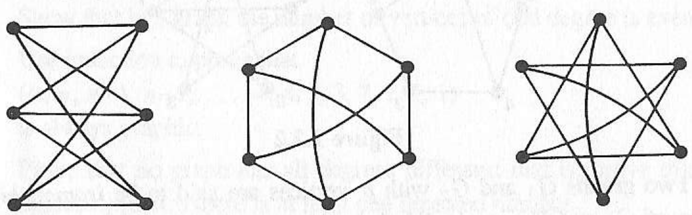


Figure 1.2.4

The first graph on the left in Figure 1.2.4 is a graph called $K_{3,3}$. In general, the graph $K_{m,n}$ is the *complete bipartite graph*, which means that $K_{m,n}$

has $m+n$ vertices divided into two sets, say m red vertices and n blue vertices. Every red vertex is adjacent to every blue vertex, and no two vertices of the same color are adjacent. Figure 1.2.5 has two different pictures of $K_{4,4}$.

K_4 is the same as the graph consisting of the vertices and edges of the tetrahedron. If we consider the vertices and edges of the cube, we get the graph called Q_3. Likewise, there are an octahedral graph, a dodecahedral graph, and an icosahedral graph. (See Figure 1.2.5).

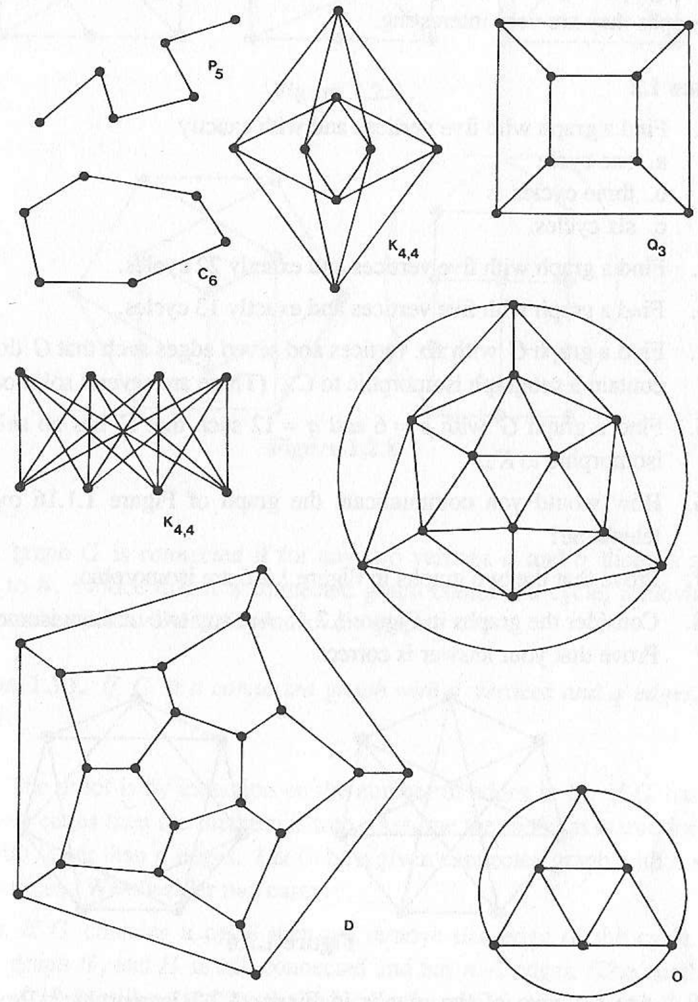


Figure 1.2.5

The graph with $n+1$ vertices labeled $x_0, x_1, \ldots, x_n$ and edges x_0x_1, x_1x_2, $x_2x_3, \ldots, x_{n-1}x_n$ is called a *path of length n*, denoted P_n. We call x_0 and x_n the *end vertices* of the path, and we say the vertices x_0 and x_n are *connected by the path* P_n. In Fig. 1.2.5 there is a picture of P_5.

The *cycle of length n*, C_n, is the graph with n vertices $x_0, x_1, \ldots, x_{n-1}$ and the edges $x_0x_1, x_1x_2, x_2x_3, \ldots, x_{n-1}x_0$. Fig. 1.2.5 shows a picture of C_6.

The graphs P_n and C_n as graphs are not so interesting, but as subgraphs of larger graphs they are very interesting.

Exercises 1.2

1.2.1. Find a graph with five vertices and with exactly
   a. one cycle
   b. three cycles
   c. six cycles.

1.2.2. Find a graph with five vertices and exactly 22 cycles.

1.2.3. Find a graph with five vertices and exactly 13 cycles.

1.2.4. Find a graph G with six vertices and seven edges such that G does not contain a subgraph isomorphic to C_4. (There are several solutions.)

1.2.5. Find a graph G with $p = 6$ and $q = 12$ such that G has no subgraph isomorphic to K_4.

1.2.6. How would you communicate the graph of Figure 1.1.16 over the telephone?

1.2.7. Prove that the two graphs in Figure 1.2.6 are isomorphic.

1.2.8. Consider the graphs in Figure 1.2.4. Are any two of them isomorphic? Prove that your answer is correct.

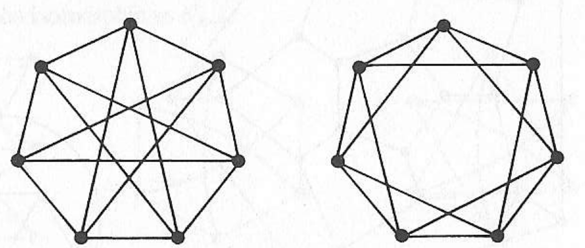


Figure 1.2.6

1.2.9. Are any two of the graphs in Figure 1.2.7 isomorphic? Prove that your answer is correct.

1.2.10. Prove that the graphs of Figures 1.1.13 and 1.2.8 are isomorphic.

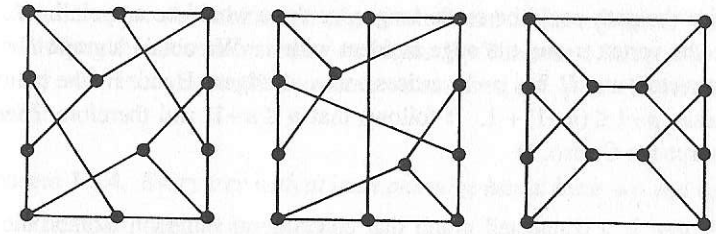


Figure 1.2.7

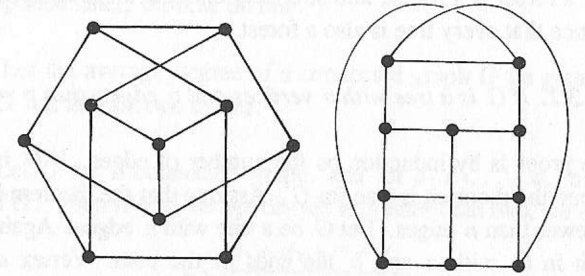


Figure 1.2.8

1.3 Trees

A graph G is *connected* if for any two vertices a and b there is a path from a to b. Notice that if a connected graph contains a cycle, removing an edge from the cycle will not disconnect the graph.

Theorem 1.3.1. *If G is a connected graph with p vertices and q edges, then* $p \leq q+1$.

Proof. The proof is by induction on the number of edges in G. If G has only one or two edges then the theorem is true. Assume the theorem is true for each graph with fewer than n edges. Let G be a given connected graph with n edges and p vertices. We consider two cases.

1) If G contains a cycle then we remove one edge of the cycle. We obtain a graph H, and H is still connected and has $n-1$ edges. The number of vertices of H is still p, and by the induction hypothesis $p \leq (n-1)+1$. Thus $p \leq n$, and certainly $p \leq n+1$.

2) If G does not contain a cycle, then we find a longest path in G. Let a and b be the vertices at the end of the path. The vertex a must be of degree 1; otherwise the path could be made longer, or there would be a cycle in G. We remove the vertex a and the edge incident with a. We obtain a graph H; H is still connected, and H has $p-1$ vertices and $n-1$ edges. Hence by the induction hypothesis, $p-1 \le (n-1)+1$. It follows that $p \le n+1$, and therefore Theorem 1.3.1 is true for G also. ■

A *tree* is a connected graph that contains no subgraph isomorphic to a cycle. For instance, a path of length n, P_n, is a tree. A *forest* is a graph that contains no cycle. Every connected subgraph of a forest is a tree. Every subgraph of a forest is a forest, and strangely enough, all subgraphs of trees are forests. Notice that every tree is also a forest.

Theorem 1.3.2. *If G is a tree with p vertices and q edges, then $p = q+1$.*

Proof. The proof is by induction on the number of edges. If G is a tree with one edge, then the theorem is true for G. Assume that the theorem is true for all trees with fewer than n edges. Let G be a tree with n edges. Again we select a longest path in G with a and b the ends of the path. Vertex a must have degree 1, since otherwisethe path could be made longer or there would be a cycle in G. Then we subtract vertex a from G together with the edge incident with a. We obtain a tree H with $p-1$ vertices and $n-1$ edges. By the induction hypothesis, $p-1=(n-1)+1$. It follows that $p=n+1$. So Theorem 1.3.2 is also true for G. ■

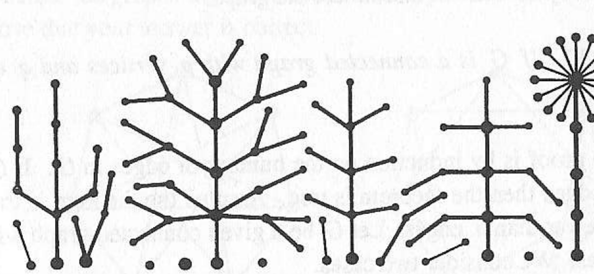


Figure 1.3.1 A forest

Theorem 1.3.3. *If G is connected, and $p = q+1$, then G is a tree.*

Proof. Assume the theorem is not true. Then there must be a connected graph G with $p=q+1$ such that G is not a tree. If G is not a tree, then G contains a cycle, and thus we can subtract an edge from G and obtain a graph H that is still connected and has p vertices and $q-1$ edges. By Theorem 1.3.1, $p \le (q-1)+1$, so $p \le q$, which is a contradiction since we assumed $p = q+1$. Therefore G is a tree. ■

Theorem 1.3.4. *Every tree with at least one edge has at least two end vertices.*

The proof is left as an exercise.

There are many interesting problems concerning trees and cycles. In order to help the reader, we now do one.

Example. Let the average degree of a connected graph G be greater than two. Prove that G has at least two cycles.

Solution. Let G be a connected graph, and let $d_1, d_2, \ldots, d_p$ be the degree sequence of G. Then if the average degree is greater than two, we have

$$2 < \frac{d_1 + d_2 + \cdots + d_p}{p}.$$

By theorem 1.1.1, the sum equal $2q$, so

$$2 < \frac{2q}{p} \quad \text{or} \quad 2p < 2q \quad \text{or} \quad p < q, \tag{1.1}$$

thus G is not a tree by Theorem 1.3.2, and hence G has at least one cycle. We subtract an edge of the cycle from G. We obtain a connected graph G' with $p' = p$ vertices and $q' = q-1$ edges. By (1.1), $p' \le q'$, and by Theorem 1.3.2 G' is not a tree, so it contains a cycle. Since we destroyed a cycle in G by removing an edge to obtain G', G must have contained at least two cycles.

Theorem 1.3.5. *A graph G is a tree if and only if there exists exactly one path between any two vertices.*

Proof. We must prove two things since the theorem says "if and only if." First we assume G is a tree. Let v_1 and v_2 be vertices of G. Since trees are connected there is a path from v_1 to v_2. Suppose there are two paths from v_1 to v_2, $P_1 = v_1u_1u_2 \cdots u_nv_2$ and $P_2 = v_1w_1w_2 \cdots w_mv_2$. If u_1 is distinct from w_1, then we follow P_1 until we find a vertex contained in P_1 that is also in P_2. (This may be v_2.) Then we have a cycle. If $u_1 = w_1$, then we look at u_2. For some i, $u_i \neq w_i$, since there are two v_1v_2 paths by assumption. Then we follow P_1 from u_{i-1} until we find a vertex contained in P_1 that is also in P_2, and then take P_2 back to u_{i-1}, and again we obtain a cycle. But G is a tree, so there are no cycles. Thus our assumption that there are two v_1v_2 paths is false.

Now we assume G is a graph with exactly one path between any two vertices. First we observe that G is connected. Suppose that G contains a cycle, $v_1v_2 \cdots v_nv_1$. There are clearly two paths from v_1 to v_n. This is a contradiction, since G has exactly one path between any two vertices. Hence G contains no cycles, and G is a tree. ■

A *spanning subgraph* of a graph G is a subgraph H of G such that H contains all the vertices of G; in other words, $V(H) = V(G)$. A *spanning tree* of G is a spanning subgraph of G that is a tree.

Theorem 1.3.6. *Every connected graph G contains a spanning tree.*

Proof. If G is a tree, there is nothing to prove, since G is then a spanning tree of G. If G is not a tree, then G contains a cycle. Let e_1 be an edge of the cycle, and let $H_1 = G - e_1$, that is, the graph obtained from G by deleting the edge e_1. If H_1 is a tree, then we are done. If not, then H_1 contains a cycle. Let e_2 be an edge of the cycle, and let $H_2 = H_1 - e_2$. We continue in this manner; since G is finite, for some i, H_i will be a tree. Thus G contains a spanning tree. ■

Exercises 1.3

1.3.1. Let T be a tree with vertices of degree only 3 or 1. If T has 10 vertices of degree 3, how many vertices of degree 1 are in T?

1.3.2. Let G be a connected graph with n vertices and n edges. How many cycles does G have?

1.3.3. Prove theorem 1.3.4.

1.3.4. Find two nonisomorphic trees that have the same degree sequence.

1.3.5. Let the average degree of the vertices of a connected graph G be less than two. How many cycles does G have?

1.3.6. Let the average degree of the vertices of a connected graph G be equal to two. How many cycles does G have?

1.3.7. The degree sequence of a tree is 5, 4, 3, 2, 1, . . . , 1. Determine the number of 1's in the sequence.

1.3.8. Show that if G is a tree, and all the degrees of vertices in G are odd, then the number of edges is odd.

1.3.9. Show by an example that the statement of Problem 1.3.8 is not true if G is not a tree.

*1.3.10. Prove that if $q > \binom{p-1}{2}$, then G is connected.

1.3.11. Prove that P_n is the only tree with $n+1$ vertices and only two vertices of degree 1.

1.3.12. Let F be a forest with 100 vertices and 90 edges. How many new edges must be added without adding vertices to obtain a tree?

1.3.13. Is there a disconnected graph with degree sequence (4, 4, 3, 3, 3, 3, 3, 3)?

1.3.14. The only connected graphs having more vertices than edges are the ____________.

1.3.15. There are only two nonisomorphic graphs with degree sequence (3, 3, 3, 3, 3, 3, 6). Find them.

1.3.16. Which of the graphs in Figure 1.3.2 is *not* isomorphic to the cube graph of Figure 1.2.5? Prove your answer.

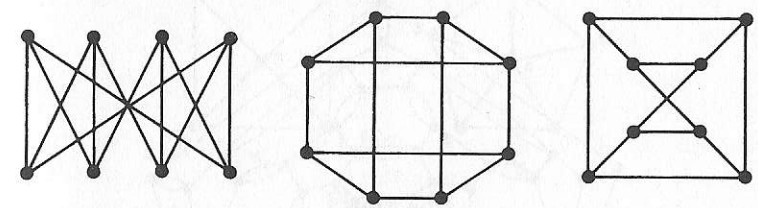


Figure 1.3.2

1.3.17. Which of the graphs in Figure 1.2.3 is isomorphic to the graph of Figure 1.3.3? Prove your answer.

1.3.18. Which of the graphs in Figure 2.1.1 is isomorphic to the graph of Figure 1.3.4? Prove your answer.

1.3.19. Which of the graphs in Figure 1.3.5 are isomorphic to graphs in Figure 1.2.4? Prove your answer.

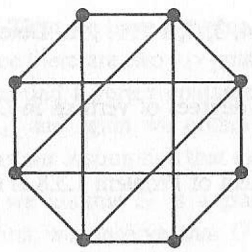


Figure 1.3.3

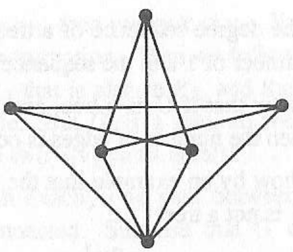


Figure 1.3.4

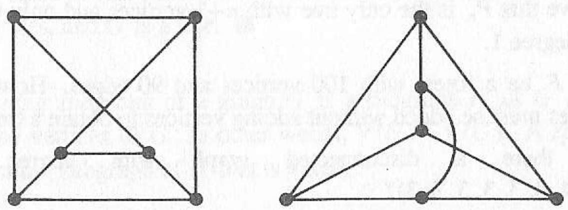


Figure 1.3.5

1.3.20 The average degree of the vertices of a tree is 1.99. How many edges does the tree have?

1.3.21 A graph with n vertices has distinct degrees except for one degree, say x, which occurs twice. Find x and prove your result. For information: If n is even, x can have two values; if n is odd x is unique.

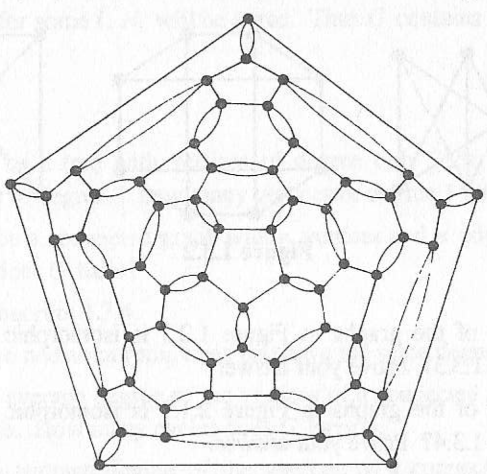


Figure 1.3.6. Buckminsterfullerene, the recently discovered soccer ball-shaped molecule with sixty carbon atoms C_{60}.

Chapter 2

COLORINGS OF GRAPHS

2.1 Vertex Colorings

The first coloring problem in graph theory is 150 years old. It is the famous four-color problem, that we shall discuss in a later chapter.

The *wheel with n spokes*, W_n, is the graph that consists of an n-cycle and one additional vertex that is adjacent to all the vertices of the cycle. W_3 is isomorphic to K_4. In Figure 2.1.1 we display W_4 and W_5.

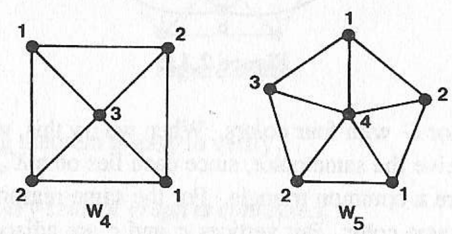


Figure 2.1.1

In Figure 2.1.1 we have assigned to each vertex a number. We shall call these numbers *colors*. Notice that no two adjacent vertices have the same color. Given a graph G, we define a *coloring of G* to be an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color.

One of the questions one can ask is, given a graph G, how many colors are necessary for a coloring of G? Clearly, if G has p vertices we can color G with p colors. The interesting question is, what is the *smallest* number of colors with which G can be colored? For a given graph G, we denote the minimum