Verifying Planarity of a Graph

Definitions:

Definition: A subdivision of an edge e is the replacement of the edge by a path of length at least two. [*Think of it as simply adding vertices in the middle of the edge e.*]
Definition: A subdivision of a graph H is the result of subdivisions of any edges of H. If G is a subdivision of H, then G is at least as large as H; H itself is also a subdivision of H.

Definition: A graph H is a minor of a graph G if H can be obtained from G by a sequence of edge deletions and/or edge contractions. [*"Minor" suggests smaller: H is smaller than G.*]
Note: If G is a subdivision of H, then H is a minor of G.

Theorems:

Theorem: (8.1.4, 8.1.6) K_5 is non-planar. $K_{3,3}$ is non-planar.
Theorem: If H is a non-planar graph, and G is a subdivision of H, then G is non-planar.
Theorem: If H is a non-planar graph, and G contains a subdivision of H (as a subgraph), then G is non-planar.
Theorem: (Kuratowski, 1930) A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.
Theorem: (Kuratowski variant) A graph is planar if and only if it contains no K_5 or $K_{3,3}$ minor.

Take-home message:

- To prove that a graph G is planar, find a planar embedding of G.
- To prove that a graph G is non-planar, either find a subgraph of G that is isomorphic to a subdivision of K_5 or $K_{3,3}$, or successively delete and contract edges of G to show that K_5 or $K_{3,3}$ is a minor of G.

Practice: Prove that for the Petersen graph P,

1. P contains a $K_{3,3}$ subdivision. [Find 6 vertices and 9 corresponding vertex-disjoint paths.]
2. P contains NO K_5 subdivision. [Show it is impossible for any subdivision of K_5 to exist as a subgraph of P.]
3. K_5 is a minor of P.

History of the Four Color Theorem

The four color conjecture took a circuitous path to becoming the four color theorem.

- 1852 – Four colorability first conjectured by Francis Guthrie, student of de Morgan

  1852–1878 de Morgan continually asks mathematicians if they know how to prove it.
- 1878/9 – Arthur Cayley presents the question to the London Math Society/publishes it.
- July 1879 – Alfred Kempe proves the Four Color Theorem, improves proof in 1880.
- 1880 – Proof by Peter Tait.
- 1890 – Percy Heawood finds an error in Cayley's proof.
- 1891 – Julius Petersen finds an error in Tait's proof. Back to conjecture status.
- 1898 – Heawood proves conjecture if every region is bounded by a multiple of three edges.
- 1922 – Franklin, using ideas of Birkhoff (1910's) proves four color conjecture for a map with ≤ 25 regions.
- 1926 – Reynolds proves four color conjecture for a map with ≤ 27 regions.
- 1940 – Winn proves four color conjecture for a map with ≤ 35 regions.
- 1970 – Ore and Stemple proves four color conjecture for a map with ≤ 39 regions.
- 1976 – Mayer proves four color conjecture for a map with ≤ 95 regions.
- 1976 – Appel and Haken used ideas of Heesch (1969) and Kempe.

  Very difficult proof, first proof relying on computers. This makes the proof controversial because the proof was unable to be hand checked.

  Idea: input 1476 graphs into a computer and show that

  - None of these graphs can appear as a subgraph in a *minimal* counterexample to the four color theorem
  - Every possible planar graph contains one of the 1476.
- Robertson, Sanders, Seymour, Thomas
  - Revision to make proof understandable
  - Reduces to 633 graphs, still uses a computer
  - `http://www.math.gatech.edu/~thomas/FC/fourcolor.html`