Vectors

We will be using vectors and matrices to store and manipulate data.

**Definition:** A vector $\vec{v}$ is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other 1-D variables. We refer to the entries of a vector by using subscripts. The length of a vector is the number of entries it has. (normally $n$)

**Example.**

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \ 2 \ 3]^T.$$

**Example.** Use a vector to represent the age distribution of a population: let $F_i$ be the number of females with ages in the interval $[5i, 5(i + 1))$. We can represent the total female population by the vector $\vec{F}$.

The females from 0 up to 5 are counted in $F_0$; those from 5 up to 10 are counted in $F_1$, etc.
**Definition:** A **matrix** $A$ is a two-dimensional array of numbers. A matrix with $m$ rows and $n$ columns is called an \( m \times n \) matrix.

★ Row by column — Row by column — Row by column ★

**Note:** A vector can be thought of as an $n \times 1$ matrix.

Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

**Example.** A generic $2 \times 3$ matrix has the form $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.

**Definition:** The matrix $B = \begin{bmatrix} 30 & 50 \\ 100 & 250 \end{bmatrix}$ is a **square matrix** because it has the same number of rows as columns.
**Example.** We will sometimes interpret a matrix as a transition matrix. In this case, the matrix is square (say $n \times n$), where the $n$ rows and $n$ columns correspond to certain states (situations). An entry $a_{i,j}$ represents transitioning from state $j$ to state $i$.

**Example.** In our population example, suppose we want to model people getting older, transitioning from one state (age group) to the next. We would set up a transition matrix such as:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

because everyone in the first age group will move to the second age group ($a_{2,1}$), everyone in state 2 will move to state 3 ($a_{3,2}$), etc. We will soon see why this is “backwards”.
Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, $A$ of size $m \times k$ and $B$ of size $l \times n$, we can find the product $AB$ if and only if $k$ equals $l$.

Let $A$ be an $m \times k$ matrix and $B$, $k \times n$. Then $AB$ is of size $m \times n$.

To calculate the entries of $AB$, remember: “Row by column”:

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}$$

When we write $A^2$, this means $AA$; $A^3$ means $AAA$, etc.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \circ & \circ \\ 0 & 1 & \circ \\ 0 & 0 & 1 \end{bmatrix}$$
The power of transition matrices

**Example.** Modeling a changing population using a matrix model.

Let us choose a size of age interval $\Delta=5$ years ("Delta"), and divide the female population into states:

*age distribution vector:*

State 0: ages $[0, 5)$ with $F_0 = 150$ females
State 1: ages $[5, 10)$ with $F_1 = 200$ females
State 2: ages $[10, 15)$ with $F_2 = 180$ females
State 3: ages $[15, 20)$ with $F_3 = 120$ females
State 4: ages $[20, 25)$ with $F_4 = 60$ females

Using a transition matrix, we can determine the population in 5 years:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}^1
\begin{bmatrix}
150 \\
200 \\
180 \\
120 \\
60
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
150 \\
150 \\
180 \\
120
\end{bmatrix}
$$

The transition matrix is "backwards" because:
Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born.

To take death into account, modify:

To take birth into account, modify:

The resulting transition matrix is called a **Leslie matrix**:

Let \( m_i \) be the average number of females that women in state \( i \) bear. Let \( p_i \) be the fraction of women in state \( i \) that survive to state \( i + 1 \).

Then

\[
\begin{bmatrix}
F_0(t + \Delta) \\
F_1(t + \Delta) \\
F_2(t + \Delta) \\
\vdots \\
F_{n-1}(t + \Delta)
\end{bmatrix}
= \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
p_0 & 0 & 0 & \cdots & 0 \\
0 & p_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & p_{n-2} & 0
\end{bmatrix}
\begin{bmatrix}
F_0(t) \\
F_1(t) \\
F_2(t) \\
\vdots \\
F_{n-1}(t)
\end{bmatrix}
\]

\[
\vec{F}(t + \Delta) = M\vec{F}(t)
\]
Leslie Matrices

Example. An animal population example (p. 47)
The population in three age groups, $F_0 = 80$, $F_1 = 40$, and $F_2 = 20$.

Suppose that as $\Delta$ time passes, everyone in state 2 dies, and one quarter of everyone else dies. Also suppose that the age-specific maternity rates are $m_0 = 0$, $m_1 = 1$, and $m_2 = 2$. Determine the Leslie matrix and the population distributions at times $\Delta$ and $2\Delta$.

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
80 \\
40 \\
20 \\
\end{bmatrix}
= \begin{bmatrix}
\end{bmatrix}
= \vec{F}(\Delta)
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
\end{bmatrix}
= \vec{F}(2\Delta)
\]
Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$ 

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find $a$ and $b$ so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$ 

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

(d) Show that the total population $P_n \approx P_0 2^n$.

- A Leslie matrix model is more descriptively realistic than the exponential model from Section 1.4, yet gives the same results.
- We’ve just worked with eigenvalues and eigenvectors!